

Universal and complete sets in martingale theory

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Abstract. The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any $G_{\delta\sigma}$ subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is Π_1^1 -complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes Π_1^1 and Σ_2^1 in the theory of martingales.

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1 Introduction

The reader should see [K2] for the notation used in this paper.

Definition 1.1 We say that a map $f : 2^{<\omega} \rightarrow [0, 1]$ is a **martingale** if $f(s) = \frac{f(s0) + f(s1)}{2}$ for each $s \in 2^{<\omega}$. The set of martingales is denoted by \mathcal{M} and is a compact subset of $[0, 1]^{2^{<\omega}}$ (equipped with the usual product topology).

This terminology is not the standard one, but the set \mathcal{M} can be interpreted as the set of all discrete martingales (in the classical sense) taking values in $[0, 1]$, as follows. If $s \in 2^{<\omega}$, then

$$N_s := \{\beta \in 2^\omega \mid s \subseteq \beta\}$$

is the usual basic clopen set. Let $f \in \mathcal{M}$. If $n \in \omega$, then let \mathcal{S}_n be the σ -algebra on 2^ω generated by $\{N_s \mid s \in 2^n\}$, and $f_n : 2^\omega \rightarrow [0, 1]$ be defined by $f_n(\beta) := f(\beta|n)$. Then the sequence $(f_n)_{n \in \omega}$ is a discrete martingale taking values in $[0, 1]$ with respect to the sequence of σ -algebras $(\mathcal{S}_n)_{n \in \omega}$ and the usual Lebesgue product measure λ on 2^ω . Conversely, if $(f_n)_{n \in \omega}$ is any such martingale, it can be viewed as an element of \mathcal{M} by setting $f(s) := f_{|s|}(\alpha)$ if $\alpha \in N_s$. This definition is correct because $f_{|s|}$, as a function measurable with respect to $\mathcal{S}_{|s|}$, has a constant value on N_s .

Definition 1.2 Let f be a martingale and $\beta \in 2^\omega$. The **oscillation** of f at β is the number

$$\text{osc}(f, \beta) := \inf_{N \in \omega} \sup_{p, q \geq N} |f(\beta|p) - f(\beta|q)|.$$

The **set of divergence** of f is $D(f) := \{\beta \in 2^\omega \mid \text{osc}(f, \beta) > 0\}$.

By definition, if f is a martingale, then

$$\beta \in D(f) \Leftrightarrow \exists r \in \omega \ \forall N \in \omega \ \exists p, q \geq N \ |f(\beta|p) - f(\beta|q)| > 2^{-r}.$$

This shows that $D(f) \in \Sigma_3^0$. Moreover, $D(f)$ has λ -measure zero, by Doob's convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any Σ_3^0 subset of 2^ω with λ -measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

Theorem 1.3 Let B be a subset of 2^ω . Then the following are equivalent:

- (a) B is Σ_3^0 and has λ -measure zero,
- (b) there is a martingale f with $B = D(f)$.

Definition 1.4 Let Γ be a class of subsets of Polish spaces, X, Y be Polish spaces, and $\mathcal{U} \subseteq Y \times X$.

- (a) We say that \mathcal{U} is **Y -universal for the Γ subsets of X** if $\mathcal{U} \in \Gamma(Y \times X)$ and $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$.
- (b) We say that \mathcal{U} is **uniformly Y -universal for the Γ subsets of X** if \mathcal{U} is Y -universal for the Γ subsets of X and, for each $S \in \Gamma(\omega^\omega \times X)$, there is a Borel map $b : \omega^\omega \rightarrow Y$ such that $S_\alpha = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^\omega$.

Corollary 1.5 Let \mathcal{G} be a G_δ subset of 2^ω with $\lambda(\mathcal{G}) = 0$. Then the set $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \beta \in D(f)\}$ is \mathcal{M} -universal for the Σ_3^0 subsets of \mathcal{G} .

In fact, we prove an effective and uniform version of the implication (a) \Rightarrow (b) in Theorem 1.3. In particular, we can associate, via a Borel map F , a martingale to a code α of an arbitrary G_δ subset G of \mathcal{G} (as in the previous corollary), in such a way that $G = D(F(\alpha))$. A consequence of this is the following:

Theorem 1.6 *The set \mathcal{P} of everywhere converging martingales is Π_1^1 -complete.*

These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact, \mathcal{P} is Π_1^1 -complete in a uniform way, which allows to derive some universal and complete sets for the whole projective hierarchy, in spaces of continuous functions, starting from \mathcal{P} . More precisely, let $P_1 := [0, 1]^{2^{<\omega}}$ and $C_1 := \mathcal{P}$. We define, for each natural number $n \geq 1$,

- the space $P_{n+1} := \mathcal{C}(2^\omega, P_n)$ of continuous functions from 2^ω into P_n , equipped with the topology of uniform convergence (inductively),
- $C_{n+1} := \{h \in P_{n+1} \mid \forall \beta \in 2^\omega \ h(\beta) \notin C_n\}$ (inductively),
- $U_n := \{(h, \beta) \in P_{n+1} \times 2^\omega \mid h(\beta) \in C_n\}$.

We prove the following:

Theorem 1.7 *Let $n \geq 1$ be a natural number. Then*

- (a) *the set U_n is uniformly P_{n+1} -universal for the Π_n^1 subsets of 2^ω ,*
- (b) *the set C_n is Π_n^1 -complete.*

In fact, our method is more general and works if we start with a Π_1^1 set which is complete in a uniform way.

Let f be a martingale. As $D(f)$ has λ -measure zero, we can associate to f the partial function $\psi(f)$ defined λ -almost everywhere by $\psi(f)(\beta) := \lim_{l \rightarrow \infty} f(\beta|l)$. The partial function $\psi(f)$ will be called the **associated partial function**. The martingale f is in \mathcal{P} if and only if $\psi(f)$ is total, in which case $\psi(f)$ is called the **associated function**. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

Theorem 1.8 (a) *The set of sequences of everywhere converging martingales whose associated functions converge pointwise is Π_1^1 -complete.*

(b) *The set of sequences of everywhere converging martingales whose associated functions converge pointwise to zero is Π_1^1 -complete.*

(c) *The set of sequences of everywhere converging martingales having a subsequence whose associated functions converge pointwise to zero is Σ_2^1 -complete.*

$2^{\Sigma_3^0}$ sets of measure zero

Notation. In the sequel, B will be a Borel subset of 2^ω , and M will be a λ -measurable subset of 2^ω . If $\beta \in 2^\omega$, then the **density of M at β** is the number $d(M, \beta) := \lim_{l \rightarrow \infty} \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ when it is defined. Note that $d(B, \beta) = 1$ if $\beta \in B$ and B is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

Theorem 2.1 (Lebesgue) *The equality $\lambda(M) = \lambda(\{\beta \in M \mid d(M, \beta) = 1\})$ holds for any λ -measurable subset M of 2^ω .*

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.2 (Lusin-Menchoff) *Let F be a closed subset of 2^ω , and $M \supseteq F$ be a λ -measurable subset of 2^ω such that $d(M, \beta) = 1$ for each $\beta \in F$. Then there is a closed subset C of 2^ω such that*

- (1) $F \subseteq C \subseteq M$,
- (2) $d(M, \beta) = 1$ for each $\beta \in C$,
- (3) $d(C, \beta) = 1$ for each $\beta \in F$.

Proof. If F is 2^ω , then we can take $C := F$. So we may assume that F is not 2^ω . We set $s^- := s \mid (|s| - 1)$ if $\emptyset \neq s \in 2^{<\omega}$. Note that $\neg F$ is the disjoint union of the elements of a sequence $(N_{s_n})_{n \in \omega}$, where $N_{s_n}^- \cap F \neq \emptyset$ for each $n \in \omega$. Fix $n \in \omega$. By Theorem 2.1,

$$\lambda(M \cap N_{s_n}) = \lambda(\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}).$$

The regularity of λ gives a closed subset F_n of 2^ω contained in $\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}$ such that $\lambda(F_n) \geq (1 - 2^{-n})\lambda(M \cap N_{s_n})$. We set $C := F \cup \bigcup_{n \in \omega} F_n$, which is closed since $|s_n| \rightarrow \infty$.

As Conditions (1) and (2) are clearly satisfied, pick $\beta \in F$. Note that

$$\begin{aligned} \lambda(N_{\beta|l} \setminus C) &= \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus C) \\ &\leq \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus F_n) \\ &\leq \sum_{s_n \supseteq \beta|l} 2^{-n} \lambda(M \cap N_{s_n}) + \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus M) \\ &\leq \sum_{s_n \supseteq \beta|l} 2^{-n} \lambda(N_{s_n}) + \lambda(N_{\beta|l} \setminus M). \end{aligned}$$

This implies that the limit of $\frac{\lambda(N_{\beta|l} \setminus C)}{\lambda(N_{\beta|l})}$ is zero since $d(M, \beta) = 1$. □

The next topology is considered in [Lu-Ma-Z], see Chapter 6.

Definition 2.3 *The τ -topology on 2^ω is generated by*

$$\mathcal{F} := \{M \subseteq 2^\omega \mid M \text{ is } \lambda\text{-measurable} \wedge \forall \beta \in M \ d(M, \beta) = 1\}.$$

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.

Lemma 2.4 *The family \mathcal{F} is a topology. In particular, any τ -open set is λ -measurable.*

Proof. Note first that \mathcal{F} is closed under finite intersections, so that it is a basis for the τ -topology. Indeed, let M, M' be in \mathcal{F} , and $\beta \in M \cap M'$. Then we use the facts that

$$\lambda(M \cap M' \cap N_{\beta|l}) = \lambda(M \cap N_{\beta|l}) - \lambda((M \cap N_{\beta|l}) \setminus M')$$

and $\lambda((M \cap N_{\beta|l}) \setminus M') \leq \lambda(N_{\beta|l} \setminus M')$.

Let \mathcal{H} be a subfamily of \mathcal{F} , and $H := \cup \mathcal{H}$. We claim that there is a countable subfamily \mathcal{C} of \mathcal{H} such that $m := \sup\{\lambda(\cup \mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{H} \text{ countable}\} = \lambda(\cup \mathcal{C})$. Indeed, for each $n \in \omega$ there is $\mathcal{D}_n \subseteq \mathcal{H}$ countable such that $\lambda(\cup \mathcal{D}_n) > m - 2^{-n}$, and $\mathcal{C} := \bigcup_{n \in \omega} \mathcal{D}_n$ is suitable. Let $C := \cup \mathcal{C}$.

Let $\beta \in H$, and M in \mathcal{H} with $\beta \in M$. Note that $\lambda(M \cup C) = \lambda(C)$ (consider the family $\mathcal{C} \cup \{M\}$). Thus $\lambda(M \setminus C) = 0$. As $d(M, \beta) = 1$, the equality $d(M \cap C, \beta) = 1$ holds, and $d(\neg C, \beta) = 0$. This implies that $H \setminus C$ is contained in $\{\beta \notin C \mid d(\neg C, \beta) < 1\}$, which has λ -measure zero by Theorem 2.1. Therefore $H \setminus C$ has λ -measure zero and $H = C \cup (H \setminus C)$ is λ -measurable.

Pick $\beta \in H$, and $M \in \mathcal{H}$ with $\beta \in M$. Then $d(M, \beta) = 1$, and thus $d(H, \beta) = 1$. Therefore $H \in \mathcal{F}$. This finishes the proof. \square

The next lemma is in the style of Urysohn's theorem (see [Lu] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.5 *Let C be a closed subset of 2^ω , and G be a G_δ subset of 2^ω disjoint from C such that $\lambda(G) = 0$. Then there is a τ -continuous map $h: 2^\omega \rightarrow [0, 1]$ such that $h|_C \equiv 0$ and $h|_G \equiv 1$.*

Proof. Let $(F_n)_{n \in \omega}$ be an increasing sequence of closed subsets of 2^ω with union $\neg G$ and $F_0 = C$. We first construct a sequence $(C_{\frac{1}{2^n}})_{n \in \omega}$ of closed subsets of 2^ω with $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$, $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$, and $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$ for each $\beta \in C_{\frac{1}{2^n}}$. We first apply Lemma 2.2 to $F := F_0$ and $M := \neg G$, which gives $F_0 \subseteq C_1 \subseteq \neg G$. Then, inductively, we apply Lemma 2.2 to $F := C_{\frac{1}{2^n}} \cup F_{n+1}$ and $M := \neg G$, which gives $C_{\frac{1}{2^n}} \cup F_{n+1} \subseteq C_{\frac{1}{2^{n+1}}} \subseteq \neg G$ such that $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$ for each $\beta \in C_{\frac{1}{2^n}}$.

Then we construct $C_{\frac{2k+1}{2^n}}$, for $0 < k < 2^{n-1}$ and $n \geq 2$. This will give us a family $(C_{\frac{k}{2^n}})_{n \in \omega, 0 < k \leq 2^n}$ of closed subsets of 2^ω . We want to ensure that $C_\zeta \subseteq C_{\zeta'}$ and $d(C_{\zeta'}, \beta) = 1$ for each $\beta \in C_\zeta$ if $\zeta' < \zeta$. We proceed by induction on n . We apply Lemma 2.2 to $F := C_{\frac{k+1}{2^{n-1}}}$ and $M := C_{\frac{k}{2^{n-1}}}$, which gives $C_{\frac{2k+1}{2^n}}$ such that $C_{\frac{k+1}{2^{n-1}}} \subseteq C_{\frac{2k+1}{2^n}} \subseteq C_{\frac{k}{2^{n-1}}}$, $d(C_{\frac{k}{2^{n-1}}}, \beta) = 1$ for each $\beta \in C_{\frac{2k+1}{2^n}}$, and $d(C_{\frac{2k+1}{2^n}}, \beta) = 1$ for each $\beta \in C_{\frac{k+1}{2^{n-1}}}$. This allows us to define \tilde{h} by

$$\tilde{h}(\beta) := \begin{cases} 0 & \text{if } \beta \in G, \\ \sup\{\zeta \mid \beta \in C_\zeta\} & \text{if } \beta \notin G. \end{cases}$$

It remains to see that \tilde{h} is τ -continuous (and then we will set $h(\beta) := 1 - \tilde{h}(\beta)$). So let $b \in (0, 1]$, and $\beta \in 2^\omega$ with $\tilde{h}(\beta) < b$. Note that there is $\zeta < b$ with $\tilde{h}(\beta) < \zeta$, so that $\beta \notin C_\zeta$. If $\gamma \notin C_\zeta$, then $\tilde{h}(\gamma) \leq \zeta < b$, so that $\neg C_\zeta$ is an open (and thus τ -open since the τ -topology is finer than the usual one) neighborhood of β on which $\tilde{h} < b$. In particular, \tilde{h} is Borel.

Now let $a \in [0, 1]$. It is enough to see that $B := \{\gamma \in 2^\omega \mid \tilde{h}(\gamma) > a\}$ is τ -open. So assume that $\tilde{h}(\gamma) > a$. Note that there are $\zeta > \zeta' > a$ with $\tilde{h}(\gamma) > \zeta$, so that $\gamma \in C_\zeta \subseteq C_{\zeta'} \subseteq B$. Thus $d(C_{\zeta'}, \gamma) = 1$, by construction of the family. As \tilde{h} is Borel, B is Borel, $d(B, \gamma)$ is defined and equal to 1. \square

Remark. We in fact proved that h is lower semi-continuous.

Notation. If $h : 2^\omega \rightarrow [0, 1]$ is a λ -measurable map and $s \in 2^{<\omega}$, then we set $\int_{N_s} h \, d\lambda := \frac{\int_{N_s} h \, d\lambda}{\lambda(N_s)}$.

Lemma 2.6 *Let $h : 2^\omega \rightarrow [0, 1]$ be a τ -continuous map, and $\beta \in 2^\omega$. Then*

$$\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} h \, d\lambda = h(\beta).$$

Proof. Let $\varepsilon > 0$, and $\beta \in M := h^{-1}(B(h(\beta), \varepsilon))$. Note that $d(M, \gamma) = 1$ for each $\gamma \in M$ since h is τ -continuous. As h is λ -measurable, we can write

$$\int_{N_{\beta|l}} h \, d\lambda = \int_{M \cap N_{\beta|l}} h \, d\lambda + \int_{N_{\beta|l} \setminus M} h \, d\lambda.$$

Note that $0 \leq \int_{N_{\beta|l} \setminus M} h \, d\lambda \leq \lambda(N_{\beta|l} \setminus M)$, so that $0 \leq \int_{N_{\beta|l} \setminus M} h \, d\lambda \leq \frac{\lambda(N_{\beta|l} \setminus M)}{\lambda(N_{\beta|l})} \rightarrow 0$. Similarly,

$$\int_{M \cap N_{\beta|l}} h \, d\lambda \in \left[(h(\beta) - \varepsilon) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}, (h(\beta) + \varepsilon) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})} \right],$$

and we are done since $\frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ tends to 1 as l tends to ∞ . \square

Now we come to our main lemma, inspired by Zahorski (see [Za]).

Lemma 2.7 *Let G be a G_δ subset of 2^ω with λ -measure zero. Then there is a martingale f with $G = D(f)$ and $\{osc(f, \beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1]$.*

Proof. Let $(G_n)_{n \in \omega}$ be a decreasing sequence of open subsets of 2^ω with intersection G and $G_0 = 2^\omega$.

• We construct $g_n : 2^\omega \rightarrow [0, 1]$, open subsets G_n^*, G_n^{**} of 2^ω , and a sequence $(s_j^n)_{j \in I_n}$ of pairwise incompatible finite binary sequences, by induction on $n \in \omega$, such that, if $S_n := \sum_{j \leq n} (-1)^j g_j$,

- (1) $G \subseteq G_{n+1}^* \subseteq G_n^{**} = \bigcup_{j \in I_n} N_{s_j^n} \subseteq G_n^* \subseteq G_n \wedge G_0^* = 2^\omega$,
- (2) $g_n|_G \equiv 1 \wedge g_n|_{G_n^*} \equiv 0$,
- (3) g_n is τ -continuous,
- (4) $g_{n+1} \leq g_n$,
- (5) $\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3} \lambda(N_{s_j^n})$,
- (6) $|\int_{N_{s_j^n}} S_n \, d\lambda - S_n(\beta)| < 2^{-3}$ if $\beta \in G \cap N_{s_j^n}$.

We set $g_0 \equiv 1$, $G_0^*, G_0^{**} := 2^\omega$, $I_0 := \{0\}$ and $s_0^0 := \emptyset$. Assume that our objects are constructed up to n .

We first construct an open subset G_{n+1}^* of 2^ω with $G \subseteq G_{n+1}^* \subseteq G_n^{**} \cap G_{n+1}$ such that

$$\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3} \lambda(N_{s_j^n})$$

if $j \in I_n$. For each $j \in I_n$, there is an open set O_j with $G \cap N_{s_j^n} \subseteq O_j \subseteq G_{n+1} \cap N_{s_j^n}$ such that $\lambda(O_j) < 2^{-n-3} \lambda(N_{s_j^n})$. We then set $G_{n+1}^* := \bigcup_{j \in I_n} O_j$.

We now apply Lemma 2.5 to $C := \neg G_{n+1}^*$ and G , which gives a τ -continuous map $h: 2^\omega \rightarrow [0, 1]$ with $h|_{\neg G_{n+1}^*} \equiv 0$ and $h|_G \equiv 1$. We set $g_{n+1} := \min(g_n, h)$, so that g_{n+1} satisfies (2)-(4).

By Lemma 2.6, $\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} S_{n+1} d\lambda = S_{n+1}(\beta)$ for each $\beta \in G$. This gives $l(\beta) \in \omega$ minimal with $|\int_{N_{\beta|l(\beta)}} S_{n+1} d\lambda - S_{n+1}(\beta)| < 2^{-3}$ and $N_{\beta|l(\beta)} \subseteq G_{n+1}^*$. The set G_{n+1}^{**} is the union of the $N_{\beta|l(\beta)}$'s, which defines I_{n+1} and $(s_j^{n+1})_{j \in I_{n+1}}$ ($S_{n+1}(\beta)$ is 0 if n is even and 1 otherwise when $\beta \in G$).

- We then define a partial map $f_\infty: 2^\omega \rightarrow [0, 1]$ by $f_\infty := \sum_{j \in \omega} (-1)^j g_j$. If $\beta \in G$, then $S_n(\beta)$ takes alternatively the values 1 and 0, depending on the parity of n , so that $f_\infty(\beta)$ is not defined. If $\beta \notin G$, then there is n such that $\beta \in \neg G_{n+1}^* \subseteq \neg G_{n+2}^* \subseteq \dots$. This implies that $f_\infty(\beta)$ is defined and equal to $S_n(\beta)$. As $0 \leq \sum_{p \leq q} (g_{2p} - g_{2p+1}) = S_{2q+1} \leq S_{2q} = g_0 + \sum_{1 \leq p \leq q} (g_{2p} - g_{2p-1}) \leq g_0$, f_∞ takes values in $[0, 1]$. So f_∞ is a partial λ -measurable map defined λ -almost everywhere since $\lambda(G) = 0$ (we use Lemma 2.4).

- This allows us to define $f: 2^{<\omega} \rightarrow [0, 1]$ by $f(s) := \int_{N_s} f_\infty d\lambda$. As $\lambda(N_s) = 2\lambda(N_{s\varepsilon})$ for each $\varepsilon \in 2$, $f(s) = \int_{N_s} f_\infty d\lambda = \frac{\int_{N_{s0}} f_\infty d\lambda + \int_{N_{s1}} f_\infty d\lambda}{\lambda(N_s)} = \frac{f(s0) + f(s1)}{2}$ and f is a martingale.

- If $\beta \notin G$, then there is n with $\beta \in G_n^* \setminus G_{n+1}^*$, so that $f_\infty(\beta) = S_n(\beta)$. By Lemma 2.6, $k \geq n$ implies that $\lim_{l \rightarrow \infty} \int_{N_{\beta|l}} S_{k+1} d\lambda = S_{k+1}(\beta) = S_n(\beta)$ since S_{k+1} is τ -continuous. Note that, for each $k \geq n$,

$$\begin{aligned} \left| \int_{N_{\beta|l}} (f_\infty - S_{k+1}) d\lambda \right| &\leq \lambda(G_{k+2}^* \cap N_{\beta|l}) \\ &\leq \sum_{\beta|l \subseteq s_j^{k+1}} \lambda(G_{k+2}^* \cap N_{s_j^{k+1}}) \\ &\leq \sum_{\beta|l \subseteq s_j^{k+1}} 2^{-k-4} \lambda(N_{s_j^{k+1}}) \\ &\leq \lambda(N_{\beta|l}) 2^{-k-4}. \end{aligned}$$

Moreover,

$$\begin{aligned} |f(\beta|l) - f_\infty(\beta)| &= \left| \int_{N_{\beta|l}} f_\infty d\lambda - f_\infty(\beta) \right| = \left| \int_{N_{\beta|l}} (f_\infty - S_{k+1}) d\lambda + \int_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta) \right| \\ &\leq 2^{-k-4} + \left| \int_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta) \right|, \end{aligned}$$

so that $\lim_{l \rightarrow \infty} f(\beta|l) = f_\infty(\beta)$, $\text{osc}(f, \beta) = 0$ and $\beta \notin D(f)$.

- If $\beta \in G$ and $n \in \omega$, then there is $j \in \omega$ with $\beta \in N_{s_j^n}$. Note that

$$f(s_j^n) = \int_{N_{s_j^n}} f_\infty d\lambda = \int_{N_{s_j^n}} S_n d\lambda + \int_{N_{s_j^n}} (f_\infty - S_n) d\lambda$$

and $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| \leq \lambda(G_{n+1}^* \cap N_{s_j^n}) < \frac{1}{8} \lambda(N_{s_j^n})$, so that $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| < \frac{1}{8}$. By (6), $|f(s_j^n) - S_n(\beta)| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. As $S_n(\beta)$ takes infinitely often the values 1 and 0, $\text{osc}(f, \beta) \geq \frac{1}{2}$ and $\beta \in D(f)$. \square

The main result will be a consequence of the main lemma and the following.

Lemma 2.8 Let $(f_n)_{n \in \omega}$ be a sequence of martingales such that

$$\{\text{osc}(f_n, \beta) \mid (n, \beta) \in \omega \times 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1].$$

Then there is a martingale f with $D(f) = \bigcup_{n \in \omega} D(f_n)$.

Proof. We first observe the following facts. Let $g, h: 2^{<\omega} \rightarrow \mathbb{R}$ be bounded, $\beta \in 2^\omega$ and $a \in \mathbb{R}$.

$$(1) \text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta).$$

This comes from the triangle inequality.

$$(2) \text{osc}(ag, \beta) = |a| \cdot \text{osc}(g, \beta).$$

$$(3) \text{osc}(g+h, \beta) = \text{osc}(h, \beta) \text{ if } \text{osc}(g, \beta) = 0.$$

By (1), $\text{osc}(h, \beta) \leq \text{osc}(g+h, \beta) + \text{osc}(-g, \beta) = \text{osc}(g+h, \beta) \leq \text{osc}(g, \beta) + \text{osc}(h, \beta) = \text{osc}(h, \beta)$, so that $\text{osc}(h, \beta) = \text{osc}(g+h, \beta)$.

$$(4) \text{osc}(g, \beta) \leq a \text{ if } g(\beta|l) \in [0, a] \text{ for each } l \in \omega.$$

• We set $D_n := D(f_n)$ for each $n \in \omega$, and $f := \sum_{n \in \omega} 4^{-n} f_n$. Note that f is defined and a martingale.

• If $\beta \notin \bigcup_{n \in \omega} D_n$, then $\text{osc}(f_n, \beta) = 0$ for each $n \in \omega$. In particular, $\text{osc}(4^{-n} f_n, \beta) = 0$ for each $n \in \omega$, by (2). Let $\varepsilon > 0$, and $M \in \omega$ with $\sum_{n > M} 4^{-n} \leq \varepsilon$. By (1), $\text{osc}(\sum_{n \leq M} 4^{-n} f_n, \beta) = 0$. By (3) and (4), $\text{osc}(f, \beta) = \text{osc}(\sum_{n > M} 4^{-n} f_n, \beta) \leq \sum_{n > M} 4^{-n} \leq \varepsilon$. As ε is arbitrary, $\text{osc}(f, \beta) = 0$, $\beta \notin D(f)$, which shows that $D(f) \subseteq \bigcup_{n \in \omega} D_n$.

• If $\beta \in \bigcup_{n \in \omega} D_n$, then let m be minimal such that $\beta \in D_m$. Note that

$$f = \sum_{n < m} 4^{-n} f_n + 4^{-m} f_m + \sum_{n > m} 4^{-n} f_n.$$

By (2) and (3), $\text{osc}(f, \beta) = \text{osc}(4^{-m} f_m + \sum_{n > m} 4^{-n} f_n, \beta)$. By (1), (2) and (4),

$$\text{osc}(f, \beta) \geq \text{osc}(4^{-m} f_m, \beta) - \text{osc}(\sum_{n > m} 4^{-n} f_n, \beta) \geq 4^{-m} \frac{1}{2} - 4^{-m} \frac{1}{3} > 0.$$

Thus $\beta \in D(f)$. □

3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.

• Let $(p_n)_{n \in \omega}$ be the sequence of prime numbers $2, 3, \dots$

• If $l \in \omega$ and $s \in \omega^l$, then $\overline{s} := \langle s(0), \dots, s(l-1) \rangle := p_0^{s(0)+1} \dots p_{l-1}^{s(l-1)+1} \in \omega$ codes s (if $l = 0$, then $\langle \rangle := 1$).

• If $\alpha \in \omega^\omega$ and $l \in \omega$, then $\overline{\alpha}(l) := \langle \alpha(0), \dots, \alpha(l-1) \rangle \in \omega$ codes $\alpha|l \in \omega^l$, and α^* is defined by removing the first coordinate: $\alpha^* := (\alpha(1), \alpha(2), \dots)$.

- If $\kappa \in \{2, \omega\}$, then $\langle \cdot, \cdot \rangle: (\kappa^\omega)^2 \rightarrow \kappa^\omega$ is a recursive homeomorphism with inverse map $\alpha \mapsto ((\alpha)_0, (\alpha)_1)$ defined for example by $(\alpha)_\varepsilon(n) := \alpha(2n + \varepsilon)$ if $(n, \varepsilon) \in \omega \times 2$ (we will also consider recursive homeomorphisms $\langle \cdot, \cdot, \cdot \rangle: (\kappa^\omega)^3 \rightarrow \kappa^\omega$ and $\langle \cdot, \cdot, \dots \rangle: (\kappa^\omega)^\omega \rightarrow \kappa^\omega$).

- If $u \in \omega$, then $\text{Seq}(u)$ means that there are $l \in \omega$ and $s \in \omega^l$ (denoted by $s(u)$) such that $u = \langle s(0), \dots, s(l-1) \rangle$. The natural number $(u)_i$ is $s(i)$ if $i < l$, and 0 otherwise. The number l is the **length** of u and is denoted by $\text{lh}(u)$. If $k \leq l$, then $\underline{u}(k) := \langle s(0), \dots, s(k-1) \rangle$, so that $\underline{u}(l) = u$. The standard basic clopen set is $N^u := \{\beta \in 2^\omega \mid \forall i < \text{lh}(u) \ \beta(i) = (u)_i\}$. We set $u^- := \langle (u)_0, \dots, (u)_{\text{lh}(u)-2} \rangle$ ($u^- := \langle \rangle$ if $\text{lh}(u) \leq 1$).

- Let X be a recursively presented Polish space. Then we will consider the effective basic open set $N(X, u) = B_X(r((u)_1)_0, \frac{((u)_1)_1}{((u)_1)_2 + 1})$.

- Let $n \geq 1$ be a natural number. A subset T of ω^n is a **tree** if $\text{Seq}(u_i)$ and $\text{lh}(u_i) = \text{lh}(u_0)$ for each $(u_0, \dots, u_{n-1}) \in T$ and each $i < n$, and $(\underline{u}_0(k), \dots, \underline{u}_{n-1}(k)) \in T$ if $(u_0, \dots, u_{n-1}) \in T$ and $k \leq \text{lh}(u_0)$.

- The next result is a part of 4A.1 in [M].

Theorem 3.1 *Let $m \geq 1$ be a natural number, and $B \in \Sigma_1^0(\omega^\omega \times (\omega^\omega)^m)$. Then there is a recursive subset T of $\omega^\omega \times \omega^m$ such that $(\alpha, \alpha_1, \dots, \alpha_m) \in B \Leftrightarrow \exists l \in \omega \ (\alpha, \underline{\alpha}_1(l), \dots, \underline{\alpha}_m(l)) \notin T$, and $T_\alpha := \{(u_0, \dots, u_{m-1}) \in \omega^m \mid (\alpha, u_0, \dots, u_{m-1}) \in T\}$ is a tree for each $\alpha \in \omega^\omega$.*

- The next result is a part of 4A.7 in [M].

Theorem 3.2 *Let X be a recursively presented Polish space and $B \in \Delta_1^1(X)$. Then we can find a recursive function $\pi: \omega^\omega \rightarrow X$ and $C \in \Pi_1^0(\omega^\omega)$ such that π is injective on C and $\pi[C] = B$.*

- We then recall some material from [L].

Notation. Let X be a recursively presented Polish space. Recall that there is a pair $(\mathcal{W}^X, \mathcal{C}^X)$ such that

- $\mathcal{W}^X \subseteq \omega$ is a Π_1^1 set of codes for the Δ_1^1 subsets of X ,
- $\mathcal{C}^X \subseteq \omega \times X$ is Π_1^1 and $\Delta_1^1(X) = \{\mathcal{C}_n^X \mid n \in \mathcal{W}^X\}$, which means that \mathcal{C}^X is “universal” for the Δ_1^1 subsets of X ,
- the relation “ $n \in \mathcal{W}^X \wedge (n, x) \notin \mathcal{C}^X$ ” is Π_1^1 in (n, x) .

If $X = \omega^\omega \times 2^\omega$, then we simply write $(\mathcal{W}, \mathcal{C}) := (\mathcal{W}^X, \mathcal{C}^X)$.

The next result will be extremely useful in the sequel.

The uniformization lemma. *Let X, Y be recursively presented Polish spaces, and $P \in \Pi_1^1(X \times Y)$. Then the set $P^+ := \{x \in X \mid \exists y \in \Delta_1^1(x) \ (x, y) \in P\}$ is Π_1^1 , and there is a partial Π_1^1 -recursive map $f: X \rightarrow Y$ such that $(x, f(x)) \in P$ for each $x \in P^+$. If moreover $S \subseteq P^+$ is a Σ_1^1 subset of X , then there is a total Δ_1^1 -recursive map $g: X \rightarrow Y$ such that $(x, g(x)) \in P$ for each $x \in S$.*

- The following definition is inspired by 3H.1 in [M].

Definition 3.3 (a) Let Γ be a class of subsets of recursively presented Polish spaces, and $\mathbf{\Gamma}$ be the associated boldface class. A system of sets $\mathcal{U}^X \in \Gamma(\omega^\omega \times X)$, where X is a recursively presented Polish space, is a **nice parametrization** in Γ for $\mathbf{\Gamma}$ if the following hold:

$$(1) \Gamma(X) = \{\mathcal{U}_\alpha^X \mid \alpha \in \omega^\omega\},$$

$$(2) \Gamma(X) = \{\mathcal{U}_\alpha^X \mid \alpha \in \omega^\omega \text{ recursive}\},$$

(3) if X is a recursively presented Polish space, then there is $\mathcal{R} : \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$ recursive such that $(\alpha, \gamma, x) \in \mathcal{U}^{\omega^\omega \times X} \Leftrightarrow (\mathcal{R}(\alpha, \gamma), x) \in \mathcal{U}^X$ if $(\alpha, \gamma, x) \in \omega^\omega \times \omega^\omega \times X$.

(b) If \mathcal{U} belongs to a nice parametrization, then we will say that \mathcal{U} is a **good universal set**.

(c) If \mathcal{U} satisfies all these properties except maybe (3), then we will say that \mathcal{U} is a **suitable universal set**.

By 3E.2, 3F.6 and 3H.1 in [M], there is a nice parametrization in Π_n^1 for $\mathbf{\Pi}_n^1$, for each natural number $n \geq 1$.

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

Theorem 3.4 (Tanaka) Let $U \in \Sigma_1^1(\omega^\omega \times \omega^\omega)$ be ω^ω -universal for the analytic subsets of ω^ω . Then $L(U) := \{(\alpha, p) \in \omega^\omega \times \omega \mid \lambda(U_\alpha \cap 2^\omega) > \frac{(p)_0}{(p)_1+1}\}$ is Σ_1^1 .

Corollary 3.5 Let $B \in \Delta_1^1(\omega^\omega \times 2^\omega)$.

(a) The map $\lambda_B : \omega^\omega \rightarrow \mathbb{R}$ defined by $\lambda_B(\alpha) := \lambda(B_\alpha)$ is Δ_1^1 -recursive, and the partial function $(n, \alpha) \mapsto \lambda(\mathcal{C}_{n,\alpha})$ is Π_1^1 -recursive on its domain $\mathcal{W} \times \omega^\omega$.

(b) Let $D \subseteq \omega$, $O_0 \in \Sigma_1^1(\omega \times \omega^\omega \times 2^\omega)$, and $O_1 \in \Pi_1^1(\omega \times \omega^\omega \times 2^\omega)$ be such that $\lambda((O_0)_{n,\alpha}) = \lambda((O_1)_{n,\alpha})$ if $n \in D$. Then the partial map $\lambda_O : D \times \omega^\omega \rightarrow \mathbb{R}$ defined by $\lambda_O(n, \alpha) := \lambda((O_0)_{n,\alpha})$ is Σ_1^1 -recursive and Π_1^1 -recursive on its domain.

(c) The partial map $d_B : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ defined by $d_B(\alpha, \beta) := d(B_\alpha, \beta)$ is Δ_1^1 -recursive, and the partial map $(n, \alpha, \beta) \mapsto d(\mathcal{C}_{n,\alpha}, \beta)$ is Π_1^1 -recursive on its Π_1^1 domain

$$\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(\mathcal{C}_{n,\alpha}, \beta) \text{ exists}\}.$$

(d) Let $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ be Δ_1^1 -recursive taking values in $[0, 1]$. Then the partial map $i_h : \omega^\omega \times \omega \rightarrow \mathbb{R}$ defined by $i_h(\alpha, u) := \int_{N^u} h(\alpha, \cdot) d\lambda$ is Δ_1^1 -recursive on its Δ_1^0 domain $\omega^\omega \times \{u \in \omega \mid \text{Seq}(u)\}$.

Proof. (a) It is enough to see that the relations $P_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) > r_p := (-1)^{(p)_0} \cdot \frac{(p)_1}{(p)_2+1}$ and

$$Q_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) < r_p$$

are Δ_1^1 to see that λ_B is Δ_1^1 -recursive. Note that there is $\phi : \omega^2 \rightarrow \omega$ recursive with $r_{\phi(p,l)} = r_p - \frac{1}{l+1}$. Thus

$$\begin{aligned} Q_B(\alpha, p) &\Leftrightarrow \exists l \in \omega \quad \lambda(B_\alpha) \leq r_p - \frac{1}{l+1} \\ &\Leftrightarrow \exists l \in \omega \quad \neg(\lambda(B_\alpha) > r_p - \frac{1}{l+1}) \\ &\Leftrightarrow \exists l \in \omega \quad \neg P_B(\alpha, \phi(p, l)), \end{aligned}$$

so that it is enough to see that P_B is Δ_1^1 .

- Now let $S \in \Sigma_1^1(\omega^\omega \times (\omega^\omega)^2)$ be a good ω^ω -universal for the analytic subsets of $(\omega^\omega)^2$. We set

$$U(\alpha, \gamma) \Leftrightarrow S((\alpha)_0, (\alpha)_1, \gamma),$$

so that $U \in \Sigma_1^1(\omega^\omega \times \omega^\omega)$ is ω^ω -universal for the analytic subsets of ω^ω . Let A be a Σ_1^1 subset of $\omega^\omega \times 2^\omega$. Then there is $\alpha_0 \in \omega^\omega$ recursive with $A = S_{\alpha_0}$, so that

$$\gamma \in A_\alpha \Leftrightarrow (\alpha_0, \alpha, \gamma) \in S \Leftrightarrow (\langle \alpha_0, \alpha \rangle, \gamma) \in U.$$

This implies that the relation $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$, equivalent to

$$((p)_0 \text{ is odd} \wedge (p)_1 > 0) \vee ((p)_0 \text{ is even} \wedge (\langle \alpha_0, \alpha \rangle, \langle (p)_1, (p)_2 \rangle) \in L(U)),$$

is Σ_1^1 , by Theorem 3.4.

- In particular, this applies to $A := B$, so that P_B is Σ_1^1 . Now note that

$$P_B(\alpha, p) \Leftrightarrow \lambda((\neg B)_\alpha) < 1 - r_p \Leftrightarrow Q_{\neg B}(\alpha, \phi'(p)),$$

for some $\phi' : \omega \rightarrow \omega$ is recursive, so that P_B is Π_1^1 by the previous computation.

- We set $C' := \{(\gamma, \beta) \in \omega^\omega \times 2^\omega \mid \gamma(0) \in \mathcal{W} \wedge (\gamma(0), \gamma^*, \beta) \in \mathcal{C}\}$. As C' is Π_1^1 ,

$$A := \{(\alpha, p) \in \omega^\omega \times \omega \mid \lambda((\neg C')_\alpha) > r_p\}$$

is Σ_1^1 , by the previous discussion. So let $n \in \mathcal{W}$. Note that

$$\begin{aligned} \lambda(\mathcal{C}_{n,\alpha}) > r_p &\Leftrightarrow \lambda(\neg \mathcal{C}_{n,\alpha}) < 1 - r_p \Leftrightarrow \lambda((\neg C')_{n\alpha}) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \quad \lambda((\neg C')_{n\alpha}) \leq 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega \quad (n\alpha, \phi''(p, l)) \notin A, \end{aligned}$$

for some recursive $\phi'' : \omega^2 \rightarrow \omega$. Similarly, the relation “ $\lambda(\mathcal{C}_{n,\alpha}) < r_p$ ” is Π_1^1 in (n, α, p) since the relation “ $n \in \mathcal{W} \wedge (n, \alpha, \beta) \notin \mathcal{C}$ ” is Π_1^1 , so that $(n, \alpha) \mapsto \lambda(\mathcal{C}_{n,\alpha})$ is Π_1^1 -recursive on $\mathcal{W} \times \omega^\omega$.

(b) Let $A := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid (\alpha(0), \alpha^*, \beta) \in O_0\}$. Note that A is Σ_1^1 . By (a), the relation $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$ is Σ_1^1 . Therefore the relation $R_{O_0}(n, \alpha, p) \Leftrightarrow R_A(n\alpha, p)$ is Σ_1^1 too. Moreover, $R_{O_0}(n, \alpha, p) \Leftrightarrow \lambda((O_0)_{n,\alpha}) > r_p \Leftrightarrow \lambda_O(n, \alpha) > r_p$.

- Assume now that $n \in D$. Then as above there is $\phi'' : \omega^2 \rightarrow \omega$ recursive such that

$$\begin{aligned} \lambda_O(n, \alpha) > r_p &\Leftrightarrow \lambda((O_1)_{n,\alpha}) > r_p \Leftrightarrow \lambda((\neg O_1)_{n,\alpha}) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \quad \lambda((\neg O_1)_{n,\alpha}) \leq 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega \quad \neg \left(\lambda((\neg O_1)_{n,\alpha}) > r_{\phi''(p, l)} \right) \\ &\Leftrightarrow \exists l \in \omega \quad \neg R_{\neg O_1}(n, \alpha, \phi''(p, l)), \end{aligned}$$

which shows the existence of $R'_{O_0} \in \Pi_1^1$ such that $\lambda_O(n, \alpha) > r_p \Leftrightarrow R'_{O_0}(n, \alpha, p)$ if $n \in D$.

- Assume that $n \in D$. Then there is $\phi' : \omega \rightarrow \omega$ recursive such that

$$\lambda_O(n, \alpha) < r_q \Leftrightarrow \lambda((O_1)_{n,\alpha}) < r_q \Leftrightarrow \lambda((\neg O_1)_{n,\alpha}) > 1 - r_q \Leftrightarrow R_{\neg O_1}(n, \alpha, \phi'(q)),$$

which shows the existence of $R''_{O_0} \in \Sigma_1^1$ such that $\lambda_O(n, \alpha) < r_q \Leftrightarrow R''_{O_0}(n, \alpha, q)$ if $n \in D$.

- Assume that $n \in D$. Then there is $\phi'' : \omega^2 \rightarrow \omega$ recursive such that

$$\begin{aligned} \lambda_O(n, \alpha) < r_q &\Leftrightarrow \lambda((O_0)_{n, \alpha}) < r_q \Leftrightarrow \exists l \in \omega \quad \lambda((O_0)_{n, \alpha}) \leq 1 - r_q - \frac{1}{l+1} \\ &\Leftrightarrow \exists l \in \omega \quad \neg \left(\lambda((O_0)_{n, \alpha}) > r_{\phi''(q, l)} \right) \Leftrightarrow \exists l \in \omega \quad \neg R_{O_0}(n, \alpha, \phi''(q, l)), \end{aligned}$$

which shows the existence of $R'''_{O_0} \in \Pi_1^1$ such that $\lambda_O(n, \alpha) < r_q \Leftrightarrow R'''_{O_0}(n, \alpha, q)$ if $n \in D$.

- Finally, $r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R_{O_0}(n, \alpha, p) \wedge R''_{O_0}(n, \alpha, q)$ and

$$r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R'_{O_0}(n, \alpha, p) \wedge R'''_{O_0}(n, \alpha, q)$$

if $n \in D$, which shows that λ_O is Σ_1^1 -recursive and Π_1^1 -recursive on $D \times \omega$.

(c) We first prove the following. Let X, Y be a recursively presented Polish spaces and $g : X \times \omega \rightarrow Y$ be a Δ_1^1 -recursive map. Then the partial map $h : X \rightarrow Y$ defined by

$$h(x) := \lim_{l \rightarrow \infty} g(x, l)$$

when this limit exists is Δ_1^1 -recursive.

Indeed, the domain D of h is $\{x \in X \mid \forall r \in \omega \quad \exists L \in \omega \quad \forall k, l \geq L \quad d_Y(g(x, k), g(x, l)) < 2^{-r}\}$, so that D is Δ_1^1 . If $x \in D$, then $h(x) \in N(Y, u)$ is equivalent to

$$\exists p, q \in \omega \quad \frac{p}{q+1} < \frac{((u)_1)_1}{((u)_1)_2 + 1} \wedge \exists L \in \omega \quad \forall l \geq L \quad g(x, l) \in N(Y, \langle 0, < ((u)_1)_0, p, q > \rangle),$$

and we are done.

- We set $B' := \{(\alpha, \gamma) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, \gamma) \in B \wedge \gamma \in N_{(\alpha)_1^* | (\alpha)_1(0)}\}$, so that $B_\alpha \cap N_{\beta|l} = B'_{<\alpha, l, \beta>}$ and B' is Δ_1^1 . By (a), the map $g : \omega^\omega \times 2^\omega \times \omega \rightarrow [0, 1]$ defined by $g(\alpha, \beta, l) := 2^{-l} \lambda(B_\alpha \cap N_{\beta|l})$ is Δ_1^1 -recursive. By the previous point, the partial map $h : \omega^\omega \times 2^\omega \rightarrow [0, 1]$ defined by

$$h(\alpha, \beta) := \lim_{l \rightarrow \infty} 2^{-l} \lambda(B_\alpha \cap N_{\beta|l})$$

when it exists is also Δ_1^1 -recursive. But $h = d_B$.

- Fix $n \in \mathcal{W}$. Then there is $q(n) \in \mathcal{W}$ such that

$$\mathcal{C}_{q(n)} = \{(\gamma, \delta) \in \omega^\omega \times 2^\omega \mid (n, (\gamma)_0, \delta) \in \mathcal{C} \wedge (\gamma)_1^* | (\gamma)_1(0) \subseteq \delta\}.$$

Moreover, we may assume that q is Π_1^1 -recursive on \mathcal{W} , by the uniformization lemma. As Π_1^1 has the substitution property, the map $g' : (n, \alpha, \beta, l) \mapsto 2^{-l} \lambda(\mathcal{C}_{q(n), <\alpha, l, \beta>}) = 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l})$ is Π_1^1 -recursive on $\mathcal{W} \times \omega^\omega \times 2^\omega \times \omega$. As above, the map

$$h' : (n, \alpha, \beta) \mapsto \lim_{l \rightarrow \infty} 2^{-l} \lambda(\mathcal{C}_{n, \alpha} \cap N_{\beta|l}) = d(\mathcal{C}_{n, \alpha}, \beta)$$

is Π_1^1 -recursive on the Π_1^1 set $\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^\omega \times 2^\omega \mid d(\mathcal{C}_{n, \alpha}, \beta) \text{ exists}\}$.

(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for $(k, l) \in \omega^2$,

$$A_{k,l} := h^{-1}\left(\left[\frac{k}{2^l}, \frac{k+1}{2^l}\right)\right)$$

and define $h_l : \omega^\omega \times 2^\omega \rightarrow [0, 1]$ by $h_l = \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}$. We also define $R \subseteq \omega^\omega \times 2^\omega \times \omega^3$ by

$$R(\alpha, \beta, u, k, l) \Leftrightarrow \frac{k}{2^l} \leq h(\alpha, \beta) < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u,$$

so that R is Δ_1^1 . Then we define $O \subseteq \omega^\omega \times 2^\omega$ by

$$O(\alpha, \beta) \Leftrightarrow \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 3 \wedge R(\alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2),$$

so that O is Δ_1^1 .

Note that (h_l) is a sequence of Borel functions pointwise converging to h . By Lebesgue's dominated convergence theorem, $\int_{N^u} h(\alpha, \cdot) d\lambda = \lim_{l \rightarrow \infty} \int_{N^u} h_l(\alpha, \cdot) d\lambda$ if $\text{Seq}(u)$. Note that

$$\begin{aligned} \int_{N^u} h_l(\alpha, \cdot) d\lambda &= \int_{N^u} \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}(\alpha, \cdot) d\lambda = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((A_{k,l})_\alpha \cap N^u) \\ &= \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(R_{\alpha, u, k, l}) = \sum_{k \leq 2^l} \frac{k}{2^l} \lambda(O_{<u, k, l>\alpha}). \end{aligned}$$

Using (a), this implies that the map $(\alpha, u, l) \mapsto \int_{N^u} h_l(\alpha, \cdot) d\lambda$ is Δ_1^1 -recursive on its Δ_1^0 domain $\omega^\omega \times \{u \in \omega \mid \text{Seq}(u)\} \times \omega$. As in the proof of (c), i_h is Δ_1^1 -recursive on its domain. \square

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).

Theorem 3.6 *Let $B \in \Delta_1^1(\omega^\omega \times 2^\omega)$, and $\epsilon : \omega^\omega \rightarrow \mathbb{R}$ be Δ_1^1 -recursive such that $\epsilon(\alpha) \in (0, 1]$ for each $\alpha \in \omega^\omega$. Then there is $T \in \Delta_1^1(\omega^\omega \times \omega)$ such that*

- (a) T_α is a tree for each $\alpha \in \omega^\omega$,
- (b) if $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$, then $K_\alpha \subseteq B_\alpha$ and $\lambda(K_\alpha) \geq \lambda(B_\alpha) - \epsilon(\alpha)$ for each $\alpha \in \omega^\omega$.

Proof. Theorem 3.2 gives $\pi : \omega^\omega \rightarrow \omega^\omega \times 2^\omega$ recursive and $C \in \Pi_1^0(\omega^\omega)$ such that π is injective on C and $\pi[C] = B$. We set $Q := \{(\alpha, \beta, \gamma) \in (\omega^\omega)^3 \mid \gamma \in C \wedge \pi(\gamma) = (\alpha, \beta)\}$. As $Q \in \Pi_1^0$, Theorem 3.1 gives a recursive subset \bar{T} of $\omega^\omega \times \omega^2$ such that $(\alpha, \beta, \gamma) \in Q \Leftrightarrow \forall l \in \omega \ (\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}$ and \bar{T}_α is a tree for each $\alpha \in \omega^\omega$.

- We set, for $u, v \in \omega$,

$$u \leq^a v \Leftrightarrow \text{Seq}(u), \text{Seq}(v) \wedge \text{lh}(u) = \text{lh}(v) \wedge \forall i < \text{lh}(u) \ (u)_i \leq (v)_i.$$

- Then we set, for $u \in \omega$ with $\text{Seq}(u)$ and $\alpha \in \omega^\omega$,

$$B_\alpha^u := \{\beta \in 2^\omega \mid \exists \gamma \in \omega^\omega \ \bar{\gamma}(\text{lh}(u)) \leq^a u \wedge \forall l \in \omega \ (\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}\}$$

and $B' := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \text{Seq}(\alpha(0)) \wedge \beta \in B_{\alpha^*}^{\alpha(0)}\}$. Note that B' is Σ_1^1 . In fact, B' is Δ_1^1 by uniqueness of the witness γ .

- We now define $\delta_\alpha \in \omega^\omega$ as follows. We define $\delta_\alpha(i)$ by induction on i . We first set

$$\delta_\alpha(0) := \min\{k \in \omega \mid \lambda(B_\alpha^{<k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2}\}.$$

This number exists since B_α is the increasing union of the $B_\alpha^{<k>}$'s. Then

$$\delta_\alpha(i+1) := \min\{k \in \omega \mid \lambda(B_\alpha^{<\delta_\alpha(0), \dots, \delta_\alpha(i), k>}) > \lambda(B_\alpha) - \frac{\epsilon(\alpha)}{2} - \dots - \frac{\epsilon(\alpha)}{2^{i+2}}\}.$$

Note that $\delta_\alpha \in \Delta_1^1(\alpha)$, by Corollary 3.5.(a).

- We set $T := \{(\alpha, v) \in \omega^\omega \times \omega \mid \text{Seq}(v) \wedge \exists u \leq^a \bar{\delta}_\alpha(\text{lh}(v)) \ (\alpha, v, u) \in \bar{T}\}$, so that $T \in \Delta_1^1(\omega^\omega \times \omega)$ and T_α is a tree for each $\alpha \in \omega^\omega$.

- We set $K := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ \beta \in B_\alpha^{\bar{\delta}_\alpha(l)}\}$, so that $K_\alpha \subseteq B_\alpha$ and

$$\lambda(K_\alpha) = \lim_{l \rightarrow \infty} \lambda(B_\alpha^{\bar{\delta}_\alpha(l)}) \geq \lambda(B_\alpha) - \epsilon(\alpha)$$

for each $\alpha \in \omega^\omega$ since $(B_\alpha^{\bar{\delta}_\alpha(l)})_{l \in \omega}$ is decreasing. It remains to apply König's lemma to see that $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$ since

$$\{s \in \omega^{<\omega} \mid \langle s(0), \dots, s(|s|-1) \rangle \leq^a \bar{\delta}_\alpha(|s|) \wedge (\alpha, \bar{\beta}(|s|), \langle s(0), \dots, s(|s|-1) \rangle) \in \bar{T}\}$$

is a finitely splitting tree. \square

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

Lemma 3.7 *Let O be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with open vertical sections. Then there is a Δ_1^1 -recursive map $f: \omega^\omega \rightarrow \omega^\omega$ such that O_α is the disjoint union $\bigcup \{N^{f(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(f(\alpha)(u))\}$, for each $\alpha \in \omega^\omega$.*

Proof. Let $P := \{(\alpha, u) \in \omega^\omega \times \omega \mid \text{Seq}(u) \wedge (\text{lh}(u) = 0 \vee (N^u \subseteq O_\alpha \wedge N^{u^-} \not\subseteq O_\alpha))\}$. Note that P is Π_1^1 , since a nonempty $\Delta_1^1(\alpha)$ closed subset of 2^ω contains a $\Delta_1^1(\alpha)$ point, by 4F.15 in [M]. We then define a relation R on $\omega^\omega \times 2^\omega \times \omega$ by $R(\alpha, \beta, u) \Leftrightarrow P(\alpha, u) \wedge \beta \in N^u$, so that R is Π_1^1 . Note that, for each $(\alpha, \beta) \in O$ there is u with $R(\alpha, \beta, u)$. By 4B.5 in [M], there is a Δ_1^1 -recursive map $g: \omega^\omega \times 2^\omega \rightarrow \omega$ such that $R(\alpha, \beta, g(\alpha, \beta))$ for each $(\alpha, \beta) \in O$. Fix $\alpha \in \omega^\omega$. Note that $S^\alpha := \{g(\alpha, \beta) \mid \beta \in O_\alpha\}$ is a $\Sigma_1^1(\alpha)$ subset of ω contained in the $\Pi_1^1(\alpha)$ set P_α . By 4B.11 and 4C in [M], there is $D^\alpha \in \Delta_1^1(\alpha)$ with $S^\alpha \subseteq D^\alpha \subseteq P_\alpha$. Note that $O_\alpha \subseteq \bigcup_{u \in D^\alpha} N^u \subseteq O_\alpha$, so that O_α is the disjoint union of the sequence $(N^u)_{u \in D^\alpha}$. We define $\delta_\alpha \in \omega^\omega$ by

$$\delta_\alpha(u) := \begin{cases} u & \text{if } u \in D_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\delta_\alpha \in \Delta_1^1(\alpha)$ and O_α is the disjoint union $\bigcup \{N^{\delta_\alpha(u)} \mid u \in \omega \wedge \text{Seq}(\delta_\alpha(u))\}$. As the set $\{(\alpha, \delta) \in \omega^\omega \times \omega^\omega \mid \delta \in \Delta_1^1(\alpha) \wedge O_\alpha \text{ is the disjoint union } \bigcup \{N^{\delta(u)} \mid u \in \omega \wedge \text{Seq}(\delta(u))\}\}$ is Π_1^1 , it remains to apply the uniformization lemma to get the desired map f . \square

Notation. We set $\mathcal{W}_1 := \{n \in \mathcal{W} \mid \forall \alpha \in \omega^\omega \exists \gamma_n \in \Delta_1^1(\alpha) \mathcal{C}_{n,\alpha} = \bigcup \{N^{\gamma_n(u)} \mid u \in \omega \wedge \text{Seq}(\gamma_n(u))\}\}$, so that, by Lemma 3.7, \mathcal{W}_1 is a Π_1^1 set of codes for the Δ_1^1 subsets of $\omega^\omega \times 2^\omega$ with open vertical sections.

Lemma 3.8 *Let F be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with closed vertical sections, and B be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ such that $B \supseteq F$ and $d(B_\alpha, \beta) = 1$ for each $(\alpha, \beta) \in F$. Then there is a Δ_1^1 subset C of $\omega^\omega \times 2^\omega$ with closed vertical sections such that*

- (1) $F \subseteq C \subseteq B$,
- (2) $d(B_\alpha, \beta) = 1$ for each $(\alpha, \beta) \in C$,
- (3) $d(C_\alpha, \beta) = 1$ for each $(\alpha, \beta) \in F$.

Proof. Lemma 3.7 gives a Δ_1^1 -recursive map $f : \omega^\omega \rightarrow \omega^\omega$ such that $(\neg F)_\alpha$ is the disjoint union $\bigcup \{N^{f(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(f(\alpha)(u))\}$, for each $\alpha \in \omega^\omega$. We set

$$B' := \left\{ (\alpha, \gamma) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, \gamma) \in B \wedge \text{Seq}\left(f((\alpha)_0)((\alpha)_1(0))\right) \wedge \gamma \in N^{f((\alpha)_0)((\alpha)_1(0))} \right\},$$

so that B' is Δ_1^1 and $B_\alpha \cap N^{f(\alpha)(u)} = B'_{<\alpha, u^\infty>}$ if $\text{Seq}(f(\alpha)(u))$. By Corollary 3.5.(c), the partial map $(\alpha, \beta, u) \mapsto d(B_\alpha \cap N^{f(\alpha)(u)}, \beta)$ is Δ_1^1 -recursive. We then set

$$B'' := \{(\alpha, \gamma) \in B' \mid d(B_{(\alpha)_0} \cap N^{f((\alpha)_0)((\alpha)_1(0))}, \gamma) = 1\},$$

so that B'' is Δ_1^1 and $\{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\} = B''_{<\alpha, u^\infty>}$ if $\text{Seq}(f(\alpha)(u))$. We define $\epsilon : \omega^\omega \rightarrow \mathbb{R}$ by

$$\epsilon(\alpha) := \begin{cases} 2^{-(\alpha)_1(0)} \lambda(B'_\alpha) & \text{if } \lambda(B'_\alpha) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

so that ϵ is Δ_1^1 -recursive by Corollary 3.5.(a), and $\epsilon(\alpha) \in (0, 1]$ for each $\alpha \in \omega^\omega$. Theorem 3.6 gives $T \in \Delta_1^1(\omega^\omega \times \omega)$ such that

- (a) T_α is a tree for each $\alpha \in \omega^\omega$,
- (b) if $K = \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall l \in \omega \ (\alpha, \bar{\beta}(l)) \in T\}$, then $K_\alpha \subseteq B''_\alpha$ and $\lambda(K_\alpha) \geq \lambda(B''_\alpha) - \epsilon(\alpha)$ for each $\alpha \in \omega^\omega$.

We set, for $u \in \omega$,

$$F^u := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \text{Seq}(f(\alpha)(u)) \wedge (<\alpha, u^\infty>, \beta) \in K \wedge \lambda(B'_{<\alpha, u^\infty>}) \neq 0\}.$$

As K is Δ_1^1 with closed vertical sections, so is F^u . If $\text{Seq}(f(\alpha)(u))$ and $\lambda(B'_{<\alpha, u^\infty>}) = 0$, then $\lambda(B_\alpha \cap N^{f(\alpha)(u)}) = 0$ and $F_\alpha^u = \emptyset$, so that $F_\alpha^u \subseteq \{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\}$ and $\lambda(F_\alpha^u) \geq (1 - 2^{-u})\lambda(B_\alpha \cap N^{f(\alpha)(u)})$. If $\text{Seq}(f(\alpha)(u))$ and $\lambda(B'_{<\alpha, u^\infty>}) \neq 0$, then

$$F_\alpha^u = K_{<\alpha, u^\infty>} \subseteq B''_{<\alpha, u^\infty>} = \{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\}.$$

Moreover,

$$\begin{aligned}\lambda(F_\alpha^u) &= \lambda(K_{<\alpha, u^\infty>}) \geq \lambda(B_{<\alpha, u^\infty>}'') - \epsilon(<\alpha, u^\infty>) = \lambda(B_{<\alpha, u^\infty>}'') - 2^{-u} \lambda(B_{<\alpha, u^\infty>}') \\ &= (1 - 2^{-u}) \lambda(B_\alpha \cap N^{f(\alpha)(u)})\end{aligned}$$

since $\lambda(B_\alpha \cap N^{f(\alpha)(u)}) = \lambda(\{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\})$, by Theorem 2.1. It remains to set $C := F \cup \bigcup_{u \in \omega} F^u$. We conclude as in the proof of Lemma 2.2. \square

- We now want to prove an effective and uniform version of Lemma 2.5.

Lemma 3.9 *Let C be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with closed vertical sections, \mathcal{G} be a Borel subset of 2^ω with $\lambda(\mathcal{G}) = 0$, and G be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with G_δ vertical sections, contained in $\omega^\omega \times \mathcal{G}$ and disjoint from C . Then there is a Δ_1^1 -recursive map $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ such that $h(\alpha, \cdot) : 2^\omega \rightarrow [0, 1]$ is τ -continuous for each $\alpha \in \omega^\omega$, $h|_C \equiv 0$ and $h|_G \equiv 1$.*

Proof. By Theorem 3.5 in [L], there is a Δ_1^1 subset F of $\omega \times \omega^\omega \times 2^\omega$ such that $F_{n,\alpha}$ is closed for each $(n, \alpha) \in \omega \times \omega^\omega$ and $\neg G = \bigcup_{n \in \omega} F_n$. Moreover, we may assume that $(F_n)_{n \in \omega}$ is increasing and $F_0 = C$.

• We will define, by primitive recursion, a partial map $f : \omega \rightarrow \omega$ which is Π_1^1 -recursive on its domain such that $f(n)$ essentially codes the set $C_{\frac{1}{2^n}}$ constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be Δ_1^1 -recursive by the uniformization lemma.

We first apply Lemma 3.8 to $F := F_0$ and $B := \neg G$. This is possible because $G_\alpha \subseteq \mathcal{G}$, so that $(\neg G)_\alpha$ has λ -measure one and therefore density one at any point of 2^ω , for each $\alpha \in \omega^\omega$. Lemma 3.8 gives $C_1 \in \Delta_1^1$ with closed vertical sections such that $\neg G \supseteq C_1 \supseteq F_0$. Let $f(0) \in \mathcal{W}_1$ with $\mathcal{C}_{f(0)} = \neg C_1$.

More generally, we will have $\mathcal{C}_{f(n)} = \neg C_{\frac{1}{2^n}}$. As mentioned above, f will be defined by primitive recursion, which means that there will be a partial map $g : \omega^2 \rightarrow \omega$ such that $f(n+1) = g(f(n), n)$. This partial map g will be Π_1^1 -recursive on its Π_1^1 domain $\{m \in \mathcal{W}_1 \mid \neg \mathcal{C}_m \subseteq \neg G\} \times \omega$, so that f will be Π_1^1 -recursive on its domain by 7A.5 in [M]. The map g will take values in \mathcal{W}_1 , and is constructed in such a way that, if $A := \neg \mathcal{C}_m \subseteq \neg G$ and $A' := \neg \mathcal{C}_{g(m,n)}$, then

- (1) $A \cup F_{n+1} \subseteq A' \subseteq \neg G$,
- (2) $\forall (\alpha, \beta) \in A' \quad d((\neg G)_\alpha, \beta) = 1$,
- (3) $\forall (\alpha, \beta) \in A \cup F_{n+1} \quad d(A'_\alpha, \beta) = 1$.

Lemma 3.8 ensures that such a $g(m, n) \in \omega$ exists if $(m, n) \in \{q \in \mathcal{W}_1 \mid \neg \mathcal{C}_q \subseteq \neg G\} \times \omega$. As the properties (1)-(3) are Π_1^1 by Corollary 3.5, the uniformization lemma ensures the existence of g . So we constructed a Δ_1^1 -recursive map $f : \omega \rightarrow \omega$, taking values in \mathcal{W}_1 , such that $C_{\frac{1}{2^n}} := \neg \mathcal{C}_{f(n)}$ is a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with closed vertical sections, $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$, $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$, and

$$d((C_{\frac{1}{2^{n+1}}})_\alpha, \beta) = 1$$

if $(\alpha, \beta) \in C_{\frac{1}{2^n}}$.

- Similarly, we construct a Δ_1^1 -recursive map $\tilde{F}:\omega \rightarrow \omega$ satisfying the following properties, if

$$D := \{p \in \omega \mid \text{Seq}(p) \wedge \text{lh}(p) = 2 \wedge 0 < (p)_1 \leq 2^{(p)_0}\}.$$

- (a) $\tilde{F}(p) \in \mathcal{W}_1$ if $p \in D$, in which case we set $C_p := \neg \mathcal{C}_{\tilde{F}(p)}$,
- (b) $C_p \subseteq C_{p'}$ if $p, p' \in D \wedge \frac{(p')_1}{2^{(p')_0}} \leq \frac{(p)_1}{2^{(p)_0}}$,
- (c) $d((C_{p'})_\alpha, \beta) = 1$ if $p, p' \in D \wedge \frac{(p')_1}{2^{(p')_0}} < \frac{(p)_1}{2^{(p)_0}} \wedge (\alpha, \beta) \in C_p$.

- This allows us to define h by

$$1 - h(\alpha, \beta) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in G, \\ \sup\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \wedge (\alpha, \beta) \in C_p\} & \text{if } (\alpha, \beta) \notin G. \end{cases}$$

Note that h is Δ_1^1 -recursive since $D \in \Delta_1^0$, so that the relation “ $p \in D \wedge (\alpha, \beta) \in C_p$ ” is Δ_1^1 in (p, α, β) . We conclude as in the proof of Lemma 2.5. \square

- We are now ready to prove the main lemma in this section. We equip the space $[0, 1]^{2^{<\omega}}$ with the distance defined by $d(f, g) := \sum_{u \in \omega, \text{Seq}(u)} \frac{|f(s(u)) - g(s(u))|}{2^{u+1}}$. We give a recursive presentation of $([0, 1]^{2^{<\omega}}, d)$. We set

$$f_n(s) := \begin{cases} \frac{((n)\bar{s})_0}{((n)\bar{s})_0 + ((n)\bar{s})_1 + 1} & \text{if } \text{Seq}(n) \wedge \forall k < \text{lh}(n) \left(\text{Seq}((n)_k) \wedge \text{lh}((n)_k) = 2 \right) \wedge \bar{s} < \text{lh}(n), \\ 0 & \text{otherwise,} \end{cases}$$

so that (f_n) is dense in $[0, 1]^{2^{<\omega}}$. It is now routine to check that the relations “ $d(f_m, f_n) \leq \frac{p}{q+1}$ ” and “ $d(f_m, f_n) < \frac{p}{q+1}$ ” are recursive in (m, n, p, q) . It is also routine to check that $F:\omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$ is Δ_1^1 -recursive if the map $F':\omega \times \omega^\omega \rightarrow \mathbb{R}$ defined by $F'(u, \alpha) := F(\alpha)(s(u))$ if $\text{Seq}(u)$, 0 otherwise, is Δ_1^1 -recursive ($s(u)$ was defined at the beginning of Section 3).

Lemma 3.10 *Let $\mathcal{V} := \{(f, \beta) \in \mathcal{M} \times 2^\omega \mid \text{osc}(f, \beta) > 0\}$, \mathcal{G} be a nonempty $G_\delta \cap \Delta_1^1$ subset of 2^ω with $\lambda(\mathcal{G}) = 0$, and G be a Δ_1^1 subset of $\omega^\omega \times 2^\omega$, contained in $\omega^\omega \times \mathcal{G}$, and with G_δ vertical sections. Then there is a Δ_1^1 -recursive map $F:\omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$, taking values in \mathcal{M} , and such that $G_\alpha = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^\omega$.*

Proof. We will define, by primitive recursion, $f:\omega \rightarrow \omega^4$ coding g_n, S_n, G_n^* , and $(s_j^n)_{j \in I_n}$ defining G_n^{**} considered in the proof of the Lemma 2.7. We must find $r:\omega^4 \times \omega \rightarrow \omega^4$ with $f(n+1) = r(f(n), n)$. In practice,

- (1) $f_0(n) \in \mathcal{W}_1$ codes $G_n^* \subseteq \omega^\omega \times 2^\omega$,
- (2) $f_1(n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ codes the graph of $g_n:\omega^\omega \times 2^\omega \rightarrow \mathbb{R}$,
- (3) $f_2(n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ codes the graph of $S_n:\omega^\omega \times 2^\omega \rightarrow \mathbb{R}$,
- (4) $f_3(n) \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$ codes the graph of the function $\alpha \mapsto (s_j^{n, \alpha})_{j \in I_{n, \alpha}}$.

- By Theorem 3.5 in [L], there is a Δ_1^1 subset O of $\omega \times \omega^\omega \times 2^\omega$ such that $O_{n, \alpha}$ is open for each $(n, \alpha) \in \omega \times \omega^\omega$ and $G = \bigcap_{n \in \omega} O_n$. Moreover, we may assume that $(O_n)_{n \in \omega}$ is decreasing and $O_0 = \omega^\omega \times 2^\omega$.

• Let $n_0 \in \mathcal{W}_1$ with $\mathcal{C}_{n_0} = \omega^\omega \times 2^\omega$, $n_1 \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ with $\mathcal{C}_{n_1}^{\omega^\omega \times 2^\omega \times \mathbb{R}} = \{(\alpha, \beta, r) \in \omega^\omega \times 2^\omega \times \mathbb{R} \mid r = 1\}$, and $n_3 \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$ with $\mathcal{C}_{n_3}^{\omega^\omega \times \omega^\omega} = \{(\alpha, \gamma) \in \omega^\omega \times \omega^\omega \mid \gamma = 10^\infty\}$. We set $f(0) := (n_0, n_1, n_1, n_3)$, so that $\mathcal{C}_{n_0} = G_0^*$, $\mathcal{C}_{n_1}^{\omega^\omega \times 2^\omega \times \mathbb{R}} = \text{Gr}(g_0) = \text{Gr}(S_0)$, $\mathcal{C}_{n_3}^{\omega^\omega \times \omega^\omega} = \text{Gr}(\alpha \mapsto 10^\infty)$,

$$\{u \in \omega \mid \text{Seq}((10^\infty)(u))\} = \{0\} = I_0$$

and $(10^\infty)(0) = 1 = < > = s_0^0$. So $f(0)$ is as desired.

• We now study the induction step. This means that we must define $r(n_0, n_1, n_2, n_3, n) \in \omega^4$.

(1) We first define $r_0(n_0, n_1, n_2, n_3, n)$ coding G_{n+1}^* . Fix $n_3 \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$ coding the graph of a Δ_1^1 -recursive function $\phi : \omega^\omega \rightarrow \omega^\omega$ such that the sequences $s(\phi(\alpha)(u))$ coded by the u 's with $\text{Seq}(\phi(\alpha)(u))$ are pairwise incompatible and $G_\alpha \subseteq \bigcup \{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))\}$ (we call P_3 the Π_1^1 set of such n_3 's). Let $\alpha \in \omega^\omega$. Assume that $\text{Seq}(\phi(\alpha)(u))$ (which intuitively means that $u \in I_{n,\alpha}$ and $s_u^{n,\alpha}$ is coded by $\phi(\alpha)(u)$). By continuity of λ ,

$$0 = \lambda(G_\alpha \cap N^{\phi(\alpha)(u)}) = \lim_{j \rightarrow \infty} \lambda(O_{j,\alpha} \cap N^{\phi(\alpha)(u)}).$$

This gives $j(n, \alpha, u) > n$ minimal with $\lambda(O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))}$ (note that $2^{-\text{lh}(\phi(\alpha)(u))} = \lambda(N^{\phi(\alpha)(u)})$). Moreover, $G_\alpha \cap N^{\phi(\alpha)(u)} \subseteq O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{n+1,\alpha} \cap N^{\phi(\alpha)(u)}$, so that $O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$ satisfies the properties of the set O_j in the proof of Lemma 2.7. We will have $G_{n+1,\alpha}^* = \bigcup_{\text{Seq}(\phi(\alpha)(u))} O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$. By Corollary 3.5 and the uniformization lemma, we may assume that the map j is Δ_1^1 -recursive on its Δ_1^1 domain

$$\{(n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid \text{Seq}(\phi(\alpha)(u))\}.$$

Note that G_{n+1}^* is a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with open vertical sections, which gives $m \in \mathcal{W}_1$ such that $\mathcal{C}_m = G_{n+1}^*$. By incompatibility, $G_{n+1,\alpha}^* \cap N^{\phi(\alpha)(u)} = O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$. So we proved that, for each $(n_3, n) \in P_3 \times \omega$, there is $m \in \mathcal{W}_1$ such that, for each $\alpha \in \omega^\omega$,

- (1) $G_\alpha \subseteq \mathcal{C}_{m,\alpha} \subseteq O_{n+1,\alpha} \cap \bigcup \{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))\}$,
- (5) $\lambda(\mathcal{C}_{m,\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\text{lh}(\phi(\alpha)(u))}$ if $u \in \omega \wedge \text{Seq}(\phi(\alpha)(u))$.

By Corollary 3.5 and the uniformization lemma, we may assume that the map $\tilde{r}_0 : (n_3, n) \mapsto m$ is Π_1^1 -recursive on $P_3 \times \omega$. We set $r_0(n_0, n_1, n_2, n_3, n) := \tilde{r}_0(n_3, n)$, which defines a partial map r_0 which is Π_1^1 -recursive on its Π_1^1 domain $\omega^3 \times P_3 \times \omega$.

(2) We now define $r_1(n_0, n_1, n_2, n_3, n)$ coding g_{n+1} . We use Lemma 3.9 and its proof. Note that $r_0(n_0, n_1, n_2, n_3, n) \in D_0 := \{m \in \mathcal{W}_1 \mid G \subseteq \mathcal{C}_m\}$. The proof of Lemma 3.9 shows that for any $m \in D_0$ there is $\tilde{F}_m \in \omega^\omega \cap \Delta_1^1$ satisfying the conditions (a), (b), (c) and

$$(d) \forall p \in D \neg(0 < (p)_1 = 2^{(p)_0}) \vee \mathcal{C}_{\tilde{F}_m(p)} \subseteq \mathcal{C}_m.$$

The uniformization lemma shows that we may assume that the partial map $\tilde{F} : m \mapsto \tilde{F}_m$ is Π_1^1 -recursive on D_0 .

The definition of h in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map $\tilde{H} : \omega \rightarrow \omega$, which is Π_1^1 -recursive on D_0 , and such that $\tilde{H}(m)$ is in $\mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}}$ and codes the graph of a Δ_1^1 -recursive map $h : \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ with

$$1 - h(\alpha, \beta) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in G \\ \sup\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \wedge (\alpha, \beta) \notin \mathcal{C}_{\tilde{H}(m)(p)}\} & \text{if } (\alpha, \beta) \notin G \end{cases}$$

if $m \in D_0$. We set $P_1 := \{c \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \mid \mathcal{C}_c \text{ is the graph of a function } \zeta_c\}$. It is routine to check that there is a Π_1^1 -recursive partial map $I : \omega^2 \rightarrow \omega$ on its domain P_1^2 such that $I(c, c') \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}}$ is the graph of the function $\min(\zeta_c, \zeta_{c'})$ if $c, c' \in P_1$. We set

$$r_1(n_0, n_1, n_2, n_3, n) := I\left(n_1, \tilde{H}(r_0(n_0, n_1, n_2, n_3, n))\right),$$

so that r_1 is Π_1^1 -recursive on its Π_1^1 domain $\omega \times P_1 \times \omega \times P_3 \times \omega$.

(3) We now define $r_2(n_0, n_1, n_2, n_3, n)$ coding

$$S_{n+1} = \begin{cases} S_n + g_{n+1} & \text{if } n \text{ is odd,} \\ S_n - g_{n+1} & \text{if } n \text{ is even.} \end{cases}$$

It is routine to check that there is a Π_1^1 -recursive partial map $S : \omega^3 \rightarrow \omega$ on its domain $P_1^2 \times \omega$ such that $S(c, c', n) \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}}$ codes the graph of the function

$$(\alpha, \beta) \mapsto \begin{cases} \zeta_c(\alpha, \beta) + \zeta_{c'}(\alpha, \beta) & \text{if } n \text{ is odd} \\ \zeta_c(\alpha, \beta) - \zeta_{c'}(\alpha, \beta) & \text{if } n \text{ is even} \end{cases}$$

if $(c, c', n) \in P_1^2 \times \omega$. We set $r_2(n_0, n_1, n_2, n_3, n) := S(n_2, r_1(n_0, n_1, n_2, n_3, n), n)$, so that r_2 is Π_1^1 -recursive on its Π_1^1 domain $\omega \times P_1^2 \times P_3 \times \omega$.

(4) We now define $r_3(n_0, n_1, n_2, n_3, n)$ coding the graph of the function $\alpha \mapsto (s_j^{n+1, \alpha})_{j \in I_{n+1, \alpha}}$. We want to ensure the two following conditions:

$$\begin{aligned} (1) \quad & G_\alpha \subseteq \bigcup_{j \in I_{n+1, \alpha}} N_{s_j^{n+1, \alpha}} \subseteq G_{n+1, \alpha}^* \\ (6) \quad & |\int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda - S_{n+1}(\alpha, \beta)| < 2^{-3} \text{ if } j \in I_{n+1, \alpha} \wedge \beta \in G_\alpha \cap N_{s_j^{n+1, \alpha}} \end{aligned}$$

Note first that in practice

$$S_{n+1}(\alpha, \beta) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

if $(\alpha, \beta) \in G$ since $g_p(\alpha, \beta) = 1$ for each p in this case. So there is $\psi : \omega \rightarrow \mathbb{R}^2$ recursive with

$$|\int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda - S_{n+1}(\alpha, \beta)| < 2^{-3} \Leftrightarrow \psi_0(n) < \int_{N_{s_j^{n+1, \alpha}}} S_{n+1}(\alpha, \cdot) d\lambda < \psi_1(n)$$

if $(\alpha, \beta) \in G$. We use Corollary 3.5 and its proof. Note that $r_2(n_0, n_1, n_2, n_3, n) \in P_1$.

We first consider $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$ (coding G_{n+1}^* and S_{n+1} respectively) as variables. We define $R_0, R_1 \subseteq \omega \times \omega^\omega \times 2^\omega \times \omega^3$ by

$$R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \exists r \in \mathbb{R} \neg (n'_2 \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}) \wedge \left(\frac{k}{2^l} \leq r < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u \right)$$

$$R_1(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow \forall r \in \mathbb{R} (n'_2 \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (n'_2, \alpha, \beta, r) \notin \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}) \vee \left(\frac{k}{2^l} \leq r < \frac{k+1}{2^l} \wedge \text{Seq}(u) \wedge \beta \in N^u \right),$$

so that R_0 is Σ_1^1 , R_1 is Π_1^1 , and $R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow R_1(n'_2, \alpha, \beta, u, k, l)$ if $n'_2 \in P_1$. Then, as in the proof of Corollary 3.5.(d), we define $O_0, O_1 \subseteq \omega \times \omega^\omega \times 2^\omega$ by

$$O_\varepsilon(n'_2, \alpha, \beta) \Leftrightarrow \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 3 \wedge R_\varepsilon(n'_2, \alpha^*, \beta, (\alpha(0))_0, (\alpha(0))_1, (\alpha(0))_2)$$

if $\varepsilon \in 2$, so that O_0 is Σ_1^1 , O_1 is Π_1^1 , and $O_0(n'_2, \alpha, \beta) \Leftrightarrow O_1(n'_2, \alpha, \beta)$ if $n'_2 \in P_1$. In particular, $n'_2 \in P_1$ and $\text{Seq}(u)$ imply that

$$\int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda = \lim_{l \rightarrow \infty} \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, < u, k, l > \alpha})$$

for each $\varepsilon \in 2$. Thus $a < \int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda < b$ is in this case equivalent to

$$\exists p_0, p_1, q_0, q_1, N \in \omega \ a < \frac{p_0}{p_1+1} \wedge \frac{q_0}{q_1+1} < b \wedge \forall l \geq N \ \frac{p_0}{p_1+1} \leq \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, < u, k, l > \alpha}) \leq \frac{q_0}{q_1+1}.$$

By Corollary 3.5.(b) applied to $D := P_1$, the partial map $\lambda_O : P_1 \times \omega^\omega \rightarrow \mathbb{R}$ defined by

$$\lambda_O(n'_2, \alpha) := \lambda((O_0)_{n'_2, \alpha})$$

is Σ_1^1 -recursive and Π_1^1 -recursive on its domain. By 3E.2, 3G.1 and 3G.2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$(n'_2, \alpha, u, l) \mapsto \sum_{k \leq 2^l} \frac{k}{2^l} \lambda((O_\varepsilon)_{n'_2, < u, k, l > \alpha})$$

is Σ_1^1 -recursive and Π_1^1 -recursive on $P_1 \times \omega^\omega \times \omega^2$. This shows the existence of $Q_0 \in \Sigma_1^1(\omega^2 \times \omega^\omega \times \omega)$ and $Q_1 \in \Pi_1^1(\omega^2 \times \omega^\omega \times \omega)$ such that

$$Q_0(n'_2, n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \Leftrightarrow \text{Seq}(u) \wedge \psi_0(n) < \int_{N^u} S_{n+1}(\alpha, \cdot) d\lambda < \psi_1(n)$$

if $n'_2 \in P_1$. We now consider $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$ as parameters. We set

$$P_{n'_0, n'_2}(n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \wedge N^u \subseteq \mathcal{C}_{n'_0, \alpha} \wedge \forall k < \text{lh}(u) \ (\neg Q_0(n'_2, n, \alpha, \underline{u}(k)) \vee N^{\underline{u}(k)} \not\subseteq \mathcal{C}_{n'_0, \alpha}).$$

Note that for each $(\alpha, \beta) \in G$ there is $l \in \omega$ minimal with the properties that $N_{\beta|l} \subseteq \mathcal{C}_{n'_0, \alpha}$ and $Q_1(n'_2, n, \alpha, < \beta(0), \dots, \beta(l-1) >)$, so that $P_{n'_0, n'_2}(n, \alpha, < \beta(0), \dots, \beta(l-1) >)$ since $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$. As $n'_0 \in \mathcal{W}_1$, $N^{\underline{u}(k)} \setminus \mathcal{C}_{n'_0, \alpha}$ is a $\Delta_1^1(\alpha)$ compact subset of 2^ω , so that it contains a $\Delta_1^1(\alpha)$ point if it is not empty (see 4F.15 in [M]). This shows that $P_{n'_0, n'_2}$ is Π_1^1 .

The uniformization lemma provides a Δ_1^1 -recursive map $L: \omega \times \omega^\omega \times 2^\omega \rightarrow \omega$ such that

$$P_{n'_0, n'_2} \left(n, \alpha, < \beta(0), \dots, \beta(L(n, \alpha, \beta) - 1) > \right)$$

if $(\alpha, \beta) \in G$. Note that the Σ_1^1 set

$$\sigma := \{ (n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid \exists \beta \in G_\alpha \ u = < \beta(0), \dots, \beta(L(n, \alpha, \beta) - 1) > \}$$

is contained in the Π_1^1 set $\pi := \{ (n, \alpha, u) \in \omega \times \omega^\omega \times \omega \mid P_{n'_0, n'_2}(n, \alpha, u) \}$. By 7B.3 in [M], there is a Δ_1^1 subset δ of $\omega \times \omega^\omega \times \omega$ such that $\sigma \subseteq \delta \subseteq \pi$. We now also consider n as a parameter and define $\varphi: \omega^\omega \rightarrow \omega^\omega$ by

$$\varphi(\alpha)(u) := \begin{cases} u & \text{if } (n, \alpha, u) \in \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Note that φ is Δ_1^1 -recursive, and that $\text{Seq}(\varphi(\alpha)(u))$ is equivalent to $(n, \alpha, u) \in \delta$. In particular,

$$\begin{aligned} (1) \quad & G_\alpha \subseteq \bigcup \{ N^{\varphi(\alpha)(u)} \mid u \in \omega \wedge \text{Seq}(\varphi(\alpha)(u)) \} \subseteq \mathcal{C}_{n'_0, \alpha} \\ (6) \quad & |f_{N^{\varphi(\alpha)(u)}} S_{n+1}(\alpha, \cdot) - S_{n+1}(\alpha, \beta)| < 2^{-3} \text{ if } \text{Seq}(\varphi(\alpha)(u)) \wedge \beta \in G_\alpha \cap N^{\varphi(\alpha)(u)} \end{aligned}$$

for each $\alpha \in \omega^\omega$. Let $k \in \mathcal{W}^{\omega^\omega \times \omega^\omega}$ such that $\mathcal{C}_k^{\omega^\omega \times \omega^\omega} = \text{Gr}(\varphi)$. We now consider n'_0, n'_2 and n as variables again. Note that for each $(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega$ there is $k \in \omega$ such that

$$R(n'_0, n'_2, n, k) \Leftrightarrow \begin{cases} k \in \mathcal{W}^{\omega^\omega \times \omega^\omega} \wedge \\ \left(\forall \alpha \in \omega^\omega \ \forall \gamma \in \omega^\omega \ (k \in \mathcal{W}^{\omega^\omega \times \omega^\omega} \wedge \neg \mathcal{C}_k^{\omega^\omega \times \omega^\omega}(k, \alpha, \gamma)) \vee \right. \\ \left. ((1) \ G_\alpha \subseteq \bigcup \{ N^{\gamma(u)} \mid u \in \omega \wedge \text{Seq}(\gamma(u)) \} \subseteq \mathcal{C}_{n'_0, \alpha} \right. \\ \left. \wedge (6) \ \forall u \in \omega \ \neg \text{Seq}(\gamma(u)) \vee Q_1(n'_2, n, \alpha, u) \right) \end{cases}$$

Note that $R \in \Pi_1^1(\omega^4)$. The uniformization lemma provides a partial map $K: \omega^3 \mapsto \omega$ which is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{W}_1 \times P_1 \times \omega$, and $R(n'_0, n'_2, n, K(n'_0, n'_2, n))$ if

$$(n'_0, n'_2, n) \in \mathcal{W}_1 \times P_1 \times \omega.$$

It remains to set $r_3(n_0, n_1, n_2, n_3, n) := K(n'_0, n'_2, n)$ if $n'_0 = r_0(n_0, n_1, n_2, n_3, n)$ and

$$n'_2 = r_2(n_0, n_1, n_2, n_3, n),$$

so that r_3 is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{W}_1 \times P_1^2 \times P_3 \times \omega$.

Finally, r is Π_1^1 -recursive on $\mathcal{W}_1 \times P_1^2 \times P_3 \times \omega$, f is Π_1^1 -recursive on ω , and thus f is Δ_1^1 -recursive by the uniformization lemma since it is total.

• We are now ready to define the dimension two versions of G_n^* , g_n , S_n , and $(s_j^n)_{j \in I_n}$:

$$\begin{aligned} (1) \quad & G_n^* := \mathcal{C}_{f_0(n)}, \\ (2) \quad & g_n(\alpha, \beta) = \rho \Leftrightarrow (f_1(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}, \\ (3) \quad & S_n(\alpha, \beta) = \rho \Leftrightarrow (f_2(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^\omega \times 2^\omega \times \mathbb{R}}, \\ (4) \quad & \begin{cases} (i) \ j \in I_{n, \alpha} \Leftrightarrow \exists \delta \in \omega^\omega \ (f_3(n), \alpha, \delta) \in \mathcal{C}^{\omega^\omega \times \omega^\omega} \wedge \text{Seq}(\delta(j)), \\ (ii) \ s_j^{n, \alpha} = \delta(j) \text{ if } j \in I_{n, \alpha}. \end{cases} \end{aligned}$$

By construction of r , these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7.

• Consequently, the martingale $F(\alpha)$ will be defined in such a way that if $u \in \omega$ codes $s \in 2^{<\omega}$, then $F(\alpha)(s) = \int_{N^u} f_\infty(\alpha, \cdot) d\lambda$. Note that $G = \bigcap_{n \in \omega} G_n^*$, so that $\neg G$ is the disjoint union of the $G_n^* \setminus G_{n+1}^*$'s. Thus

$$\begin{aligned} \int_{N^u} f_\infty(\alpha, \cdot) d\lambda &= \int_{N^u \setminus G_\alpha} f_\infty(\alpha, \cdot) d\lambda = \sum_{n \in \omega} \int_{N^u \cap (G_n^* \setminus (G_{n+1}^*)_\alpha)} f_\infty(\alpha, \cdot) d\lambda \\ &= \sum_{n \in \omega} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus (G_{n+1}^*)_\alpha)} g_j(\alpha, \cdot) d\lambda \\ &= \lim_{l \rightarrow \infty} \sum_{n \leq l} \sum_{j \leq n} (-1)^j \int_{N^u \cap (G_n^* \setminus (G_{n+1}^*)_\alpha)} g_j(\alpha, \cdot) d\lambda. \end{aligned}$$

Consequently, in order to prove that F is Δ_1^1 -recursive, it is enough to check that the partial map $(u, \alpha, j, n) \mapsto \int_{N^u \cap (G_n^* \setminus (G_{n+1}^*)_\alpha)} g_j(\alpha, \cdot) d\lambda$ is Δ_1^1 -recursive from $\{u \in \omega \mid \text{Seq}(u)\} \times \omega^\omega \times \omega^2$ into \mathbb{R} . By Corollary 3.5, it is enough to check that the map $h: \omega^\omega \times 2^\omega \rightarrow \mathbb{R}$ defined by

$$h(\alpha, \beta) := \begin{cases} g_{(\alpha(0))_0}(\alpha^*, \beta) & \text{if } \text{Seq}(\alpha(0)) \wedge \text{lh}(\alpha(0)) = 2 \wedge (\alpha^*, \beta) \in G_{(\alpha(0))_1}^* \setminus G_{(\alpha(0))_1+1}^*, \\ 0 & \text{otherwise,} \end{cases}$$

is Δ_1^1 -recursive. This comes from the facts that

$$(\alpha, \beta) \in G_n^* \Leftrightarrow (f_0(n), \alpha, \beta) \in \mathcal{C} \Leftrightarrow \neg(f_0(n) \in \mathcal{W} \wedge (f_0(n), \alpha, \beta) \notin \mathcal{C})$$

is Δ_1^1 in (α, β, n) and

$$\begin{aligned} g_n(\alpha, \beta) \in N(\mathbb{R}, p) &\Leftrightarrow \exists \rho \in \mathbb{R} \neg \left(f_1(n) \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}^{\omega \times 2^\omega \times \mathbb{R}} \right) \wedge \\ &\quad \rho \in N(\mathbb{R}, p) \\ &\Leftrightarrow \forall \rho \in \mathbb{R} \left(f_1(n) \in \mathcal{W}^{\omega \times 2^\omega \times \mathbb{R}} \wedge (f_1(n), \alpha, \beta, \rho) \notin \mathcal{C}^{\omega \times 2^\omega \times \mathbb{R}} \right) \vee \\ &\quad \rho \in N(\mathbb{R}, p) \end{aligned}$$

is Δ_1^1 in (α, β, n, p) .

• Finally, the map F is Δ_1^1 -recursive and is as required. □

4 First consequences

(A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

Theorem 4.1 (Zahorski) *Let B be a subset of $[0, 1]$. The following are equivalent:*

(a) *there are $S \in \Sigma_2^0$ and $P \in \Pi_3^0$ with $m(P) = 1$, where m is the Lebesgue measure on $[0, 1]$, such that $B = S \cap P$,*

(b) *there is $f \in C([0, 1])$ with $B = \{x \in [0, 1] \mid f'(x) \text{ exists}\}$ (we consider only one-sided derivatives at the endpoints).*

The second result is 23.23.

Theorem 4.2 *Let \mathcal{G} be a G_δ subset of $(0, 1)$ with $m(\mathcal{G})=0$. Then*

$$\{(f, x) \in C([0, 1]) \times \mathcal{G} \mid f'(x) \text{ exists}\}$$

is $C([0, 1])$ -universal for $\Pi_3^0(\mathcal{G})$.

- We prove results in that spirit here.

Theorem 4.3 *Let B be a subset of 2^ω . Then the following are equivalent:*

- (a) *B is Σ_3^0 and has λ -measure zero,*
- (b) *there is $f \in \mathcal{M}$ with $B = \{\beta \in 2^\omega \mid \text{osc}(f, \beta) > 0\}$.*

Proof. (a) \Rightarrow (b) Write $B = \bigcup_{n \in \omega} G_n$, where the G_n 's are G_δ . Lemma 2.7 gives, for each n , a martingale f_n with $G_n = D(f_n)$ and $\{\text{osc}(f_n, \beta) \mid \beta \in 2^\omega\} \subseteq \{0\} \cup [\frac{1}{2}, 1]$. Lemma 2.8 gives $f \in \mathcal{M}$ with $D(f) = B$.

(b) \Rightarrow (a) We already noticed in the introduction that B is Σ_3^0 . By Doob's theorem, B has λ -measure zero (see [D]). \square

Corollary 4.4 *Let \mathcal{G} be a G_δ subset of 2^ω with $\lambda(\mathcal{G})=0$. Then $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \text{osc}(f, \beta) > 0\}$ is \mathcal{M} -universal for $\Sigma_3^0(\mathcal{G})$.*

For example, $\{\beta \in 2^\omega \mid \forall n \in \omega \ \beta(2n)=0\}$ is a Π_1^0 copy of 2^ω and has λ -measure zero.

(B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski's theorem, which allows to prove the following result

Theorem 4.5 (Mazurkiewicz) *The set of differentiable functions in $C([0, 1])$ is Π_1^1 -complete.*

- Here again, there is a result in that spirit.

Theorem 4.6 *The set $\mathcal{P} := \{f \in \mathcal{M} \mid \forall \beta \in 2^\omega \ \text{osc}(f, \beta) = 0\}$ is Π_1^1 -complete.*

Notation. Let $\mathcal{K} := \{\beta \in 2^\omega \mid \forall n \in \omega \ \beta(2n)=0\}$, which is a Π_1^0 copy of the Cantor space 2^ω with $\lambda(\mathcal{K})=0$. In particular, \mathcal{K} is a nonempty $G_\delta \cap \Delta_1^1$ subset of 2^ω .

Proof. Let $U \in \Pi_1^1(\omega^\omega \times 2^\omega)$ be ω^ω -universal for the co-analytic subsets of 2^ω , and

$$\Pi := \{\alpha \in \omega^\omega \mid ((\alpha)_0, (\alpha)_1) \in U\}.$$

Note that $\Pi \in \Pi_1^1$. If $P \in \Pi_1^1(2^\omega)$, then $P = U_\alpha$ for some $\alpha \in \omega^\omega$, so that the map $\beta \mapsto \langle \alpha, \beta \rangle$ is a continuous reduction of P to Π and Π is Π_1^1 -complete. Let $H \in \Pi_2^0(\omega^\omega \times 2^\omega)$ with $\neg \Pi = \Pi_0[H]$. We set $G := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid (\alpha, (\beta)_1) \in H \wedge \beta \in \mathcal{K}\}$, so that $G \in \Delta_1^1(\omega^\omega \times 2^\omega)$, has G_δ vertical sections and $G \subseteq \omega^\omega \times \mathcal{K}$. Lemma 3.10 gives $F: \omega^\omega \rightarrow \mathcal{M}$ Borel such that $G_\alpha = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^\omega$.

Thus

$$\alpha \notin \Pi \Leftrightarrow \exists \beta \in 2^\omega \ (\alpha, \beta) \in H \Leftrightarrow \exists \beta \in 2^\omega \ (\alpha, \beta) \in G \Leftrightarrow \exists \beta \in 2^\omega \ (F(\alpha), \beta) \in \mathcal{V} \Leftrightarrow F(\alpha) \notin \mathcal{P}.$$

Thus $\Pi = F^{-1}(\mathcal{P})$ and \mathcal{P} is Borel Π_1^1 -complete. By 26.C in [K2], \mathcal{P} is Π_1^1 -complete. \square

- We now prove Theorem 1.8. Let X be a metrizable compact space and Y be a Polish space. We equip $\mathcal{C}(X, Y)$ with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map ψ defined before Theorem 1.8.

Theorem 4.7 (a) *The set $\mathcal{P}_1 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges}\}$ is Π_1^1 -complete.*

(b) *The set $\mathcal{P}_2 := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges to zero}\}$ is Π_1^1 -complete.*

(c) *The set $\mathcal{S} := \{(f_k)_{k \in \omega} \in \mathcal{P}^\omega \mid \exists \gamma \in \omega^\omega \ (\psi(f_{\gamma(i)}))_{i \in \omega} \text{ pointwise converges to zero}\}$ is Σ_2^1 -complete.*

Proof. We define $\varphi: \mathcal{C}(2^\omega, [0, 1]) \rightarrow \mathcal{M}$ by $\varphi(h)(s) := \int_{N_s} h \, d\lambda$. As in the proof of Lemma 2.7, φ is well-defined. It is also continuous, and injective: if $h \neq h'$, then we can find $q \in \omega$ and $s \in 2^{<\omega}$ such that $h(\beta) - h'(\beta) > 2^{-q}$ for each $\beta \in N_s$ or $h'(\beta) - h(\beta) > 2^{-q}$ for each $\beta \in N_s$, so that

$$|\varphi(h)(s) - \varphi(h')(s)| = \frac{1}{\lambda(N_s)} \left| \int_{N_s} h \, d\lambda - \int_{N_s} h' \, d\lambda \right| \geq 2^{-q}.$$

This implies that the range \mathcal{R} of φ is Borel and $\psi := \varphi^{-1}: \mathcal{R} \rightarrow \mathcal{C}(2^\omega, [0, 1])$ is Borel. As every continuous map $h: 2^\omega \rightarrow [0, 1]$ is τ -continuous,

$$\lim_{l \rightarrow \infty} \varphi(h)(\beta|l) = \lim_{l \rightarrow \infty} \int_{N_{\beta|l}} h \, d\lambda = h(\beta)$$

for each $\beta \in 2^\omega$, by Lemma 2.6. This implies that $f \in \mathcal{P}$ and $\psi(f)(\beta) = \lim_{l \rightarrow \infty} f(\beta|l)$ for each $\beta \in 2^\omega$ if $f \in \mathcal{R}$.

(a) Note that the proof of 33.11 in [K2] shows that the set

$$P_1 := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid (h_k)_{k \in \omega} \text{ pointwise converges}\}$$

is Π_1^1 -complete. As $\mathcal{E} := \{(f_k)_{k \in \omega} \in \mathcal{R}^\omega \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges}\} = (\psi^\omega)^{-1}(P_1)$, the equalities $P_1 = (\varphi^\omega)^{-1}(\mathcal{E}) = (\varphi^\omega)^{-1}(\mathcal{P}_1)$ hold and \mathcal{P}_1 is Π_1^1 -complete.

(b) We argue as in (a).

(c) As in [B-Ka-L], the set

$$S := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega \mid \exists \gamma \in \omega^\omega \ (h_{\gamma(i)})_{i \in \omega} \text{ pointwise converges to zero}\},$$

is Σ_2^1 -complete. Indeed, fix $Q \in \Sigma_2^1(2^\omega)$.

Lemma 2.2 in [B-Ka-L] gives $(g_k)_{k \in \omega} \in (\mathcal{C}(2^\omega \times 2^\omega, 2))^\omega$ such that, for each $\delta \in 2^\omega$, the following are equivalent:

- (i) $\delta \in Q$,
- (ii) $\exists \gamma \in \omega^\omega \ \forall \beta \in 2^\omega \ \lim_{i \rightarrow \infty} g_{\gamma(i)}(\delta, \beta) = 0$.

We define, $g: 2^\omega \rightarrow (\mathcal{C}(2^\omega, [0, 1]))^\omega$ by $g(\delta)(k)(\beta) := g_k(\delta, \beta)$. Then g is continuous and reduces Q to S . As

$$\mathcal{E}' := \{ (f_k)_{k \in \omega} \in \mathcal{R}^\omega \mid \exists \gamma \in \omega^\omega \ (\psi(f_{\gamma(i)}))_{i \in \omega} \text{ pointwise converges to zero} \} = (\psi^\omega)^{-1}(S),$$

$S = (\varphi^\omega)^{-1}(\mathcal{E}') = (\varphi^\omega)^{-1}(S)$ and S is Σ_2^1 -complete. \square

5 Universal and complete sets in the spaces $\mathcal{C}(2^\omega, X)$

- It is known that if Γ is a self-dual Wadge class and X is a Polish space, then there is no set which is X -universal for the subsets of X in Γ (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

Proposition 5.1 *Let X be a Polish space, Γ be a Wadge class with complete set $C \in \Gamma(X)$, and $\mathcal{U}^\Gamma := \{(h, \beta) \in \mathcal{C}(2^\omega, X) \times 2^\omega \mid h(\beta) \in C\}$. Then \mathcal{U}^Γ is $\mathcal{C}(2^\omega, X)$ -universal for the Γ subsets of 2^ω .*

Proof. As the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous, $\mathcal{U}^\Gamma \in \Gamma$. If $A \in \Gamma(2^\omega)$, then $A = h^{-1}(C)$ for some $h \in \mathcal{C}(2^\omega, X)$, so that $A = \mathcal{U}_h^\Gamma$. \square

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel Π_1^1 -complete set is actually Π_1^1 -complete. In fact, Kechris's proof shows the result for the classes Π_n^1 . Our main tool is a uniform version of this. Kechris's result has recently been strengthened in [P] as follows.

Theorem 5.2 (Pawlikowski) *Let $n \geq 1$ be a natural number, and $C \subseteq X \subseteq 2^\omega$. If Borel functions from 2^ω into X give as preimages of C all Π_n^1 subsets of 2^ω , then so do continuous injections.*

The main tool mentioned above is the following:

Theorem 5.3 *Let $n \geq 1$ be a natural number, $\mathcal{U}^{\Pi_n^1, 2^\omega}$ be a suitable ω^ω -universal set for the Π_n^1 subsets of 2^ω , X be a recursively presented Polish space, $C \in \Pi_n^1(X)$, $\mathcal{R}: \omega^\omega \times \omega^\omega \rightarrow \omega^\omega$ be a recursive map, and $b: \omega^\omega \rightarrow X$ be a Δ_1^1 -recursive map such that*

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow b(\mathcal{R}(\alpha, \beta)) \in C$$

for each $(\alpha, \beta) \in \omega^\omega \times 2^\omega$. Then there is a Δ_1^1 -recursive map $f: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X)$ such that

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow f(\alpha)(\beta) \in C$$

for each $(\alpha, \beta) \in \omega^\omega \times 2^\omega$.

- We first recall some material from [K3].

Definition 5.4 (a) A **coding system** for nonempty perfect binary trees is a pair $(\mathcal{D}, \mathcal{O})$, where $\mathcal{D} \subseteq 2^\omega$ and $\mathcal{O}: \mathcal{D} \rightarrow \{T \in 2^{2^{<\omega}} \mid T \text{ is a nonempty perfect binary tree}\}$ is onto.

(b) A coding system $(\mathcal{D}, \mathcal{O})$ is **nice** if

(i) for any $\alpha \in \omega^\omega$ and any $\Delta_1^1(\alpha)$ -recursive map $H: 2^\omega \times 2^\omega \rightarrow \omega$, we can find $\beta \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k \in \omega$ such that $H(\beta, \delta) = k$ for each δ in the body $[\mathcal{O}(\beta)]$ of $\mathcal{O}(\beta)$,

(ii) \mathcal{D} is Π_1^1 and, for $\beta \in \mathcal{D}$, the relation

$$R(m, \beta) \Leftrightarrow \text{Seq}(m) \wedge ((m)_0, \dots, (m)_{\text{lh}(m)-1}) \in \mathcal{O}(\beta)$$

is Δ_1^1 , i.e., there are Π_1^1 relations Π_0, Π_1 such that $R(m, \beta) \Leftrightarrow \Pi_0(m, \beta) \Leftrightarrow \neg \Pi_1(m, \beta)$ if $\beta \in \mathcal{D}$.

Nice coding systems exist. If $\beta \in \mathcal{D}$, then there is a canonical homeomorphism β^* from $[\mathcal{O}(\beta)]$ onto 2^ω . We now check that the construction of β^* is effective.

Lemma 5.5 (a) The partial function $e: (\beta, \delta) \mapsto \beta^*(\delta)$ is Π_1^1 -recursive on its Π_1^1 domain

$$\text{Domain}(e) := \{(\beta, \delta) \in \mathcal{D} \times 2^\omega \mid \delta \in [\mathcal{O}(\beta)]\}.$$

(b) The partial function $\iota: (\beta, \gamma) \mapsto$ the unique $\delta \in [\mathcal{O}(\beta)]$ with $\beta^*(\delta) = \gamma$ is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{D} \times 2^\omega$.

Proof. (a) We define a Π_1^1 relation \mathcal{Q} on $\omega^2 \times (2^\omega)^2$ by

$$\mathcal{Q}(p, p', \beta, \delta) \Leftrightarrow \left((\forall \varepsilon \in 2 \ \Pi_0(\overline{(\delta|p')\varepsilon}, \beta)) \wedge (\forall p \leq p'' < p' \ \exists \varepsilon \in 2 \ \Pi_1(\overline{(\delta|p'')\varepsilon}, \beta)) \right).$$

Note that

$$\beta^*(\delta)(n) = \varepsilon \Leftrightarrow \begin{cases} \exists l \in \omega \ \text{Seq}(l) \wedge \text{lh}(l) = n+1 \wedge \delta((l)_n) = \varepsilon \wedge \mathcal{Q}(0, (l)_0, \beta, \delta) \wedge \\ \forall m < n \ (l)_m < (l)_{m+1} \wedge \mathcal{Q}((l)_m+1, (l)_{m+1}, \beta, \delta) \end{cases}$$

if $\beta \in \mathcal{D}$. The proof of (b) is similar. \square

- Let X be a recursively presented Polish space, and d_X and $(r_n^X)_{n \in \omega}$ be respectively a distance function and a recursive presentation of X . We now give a **recursive presentation of $\mathcal{C}(2^\omega, X)$** , equipped with the usual distance defined by

$$d(h, h') := \sup_{\beta \in 2^\omega} d_X(h(\beta), h'(\beta)),$$

since this is not present in [M]. We define, by primitive recursion, a recursive map $\nu: \omega \rightarrow \omega$ such that $\nu(i)$ enumerates $\{s \in 2^{<\omega} \mid |s| = i\}$. We first set $\nu(0) := 1 = \langle \rangle$. Then

$$\nu(i+1) = k \Leftrightarrow \text{Seq}(k) \wedge \text{lh}(k) = 2^{i+1} \wedge \forall l < 2^i \ \forall \varepsilon \in 2 \ (k)_{\varepsilon 2^i + l} = \overline{s\left(\left(\nu(i)\right)_l\right)\varepsilon}.$$

If $\text{Seq}(n)$ and $\text{lh}(n) = 2^i$ for some $i (< n)$, then we define $h_n: 2^\omega \rightarrow X$ by $h_n(\beta) := r_{(n)_l}^X$ if

$$\beta|_i = s_l^i := s\left(\left(\nu(i)\right)_l\right).$$

If $\neg \text{Seq}(n)$ or $\text{lh}(n) \neq 2^i$ for each i , then we define $h_n: 2^\omega \rightarrow X$ by $h_n(\beta) := r_0^X$ if $\beta \in 2^\omega$. In any case, $h_n \in \mathcal{C}(2^\omega, X)$ and takes finitely many values.

Lemma 5.6 *Let X be a recursively presented Polish space. Then the sequence $(h_n)_{n \in \omega}$ is a recursive presentation of $\mathcal{C}(2^\omega, X)$, equipped with d .*

Proof. We have to see that (h_n) is dense in $\mathcal{C}(2^\omega, X)$. So let $h \in \mathcal{C}(2^\omega, X)$, $\epsilon > 0$ and $m \in \omega$ with $2^{-m} < \frac{\epsilon}{2}$. As h is uniformly continuous, there is $i \in \omega$ such that $d_X(h(\beta), h(\delta)) < 2^{-m}$ if $\beta|_i = \delta|_i$. We choose, for each $l < 2^i$, $n_l \in \omega$ such that $d_X(r_{n_l}^X, h(s_l^i 0^\infty)) < 2^{-m}$. We set $n := \langle n_0, \dots, n_{2^i-1} \rangle$. If $\beta \in 2^\omega$ and $\beta|_i = s_l^i$, then $d_X(h(\beta), h_n(\beta)) \leq d_X(h(\beta), h(s_l^i 0^\infty)) + d_X(h(s_l^i 0^\infty), r_{n_l}^X) \leq 2^{-m} + 2^{-m}$, so that $d(h, h_n) < \epsilon$. It is routine to check that the relations “ $d(h_m, h_n) \leq \frac{p}{q+1}$ ” and “ $d(h_m, h_n) < \frac{p}{q+1}$ ” are recursive in (m, n, p, q) . \square

We saw in the proof of Proposition 5.1 that the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous from $\mathcal{C}(2^\omega, X) \times 2^\omega$ into X . We can say more if X is recursively presented.

Lemma 5.7 *Let X be a recursively presented Polish space. Then the evaluation map is recursive.*

Proof. Note that

$$\begin{aligned} h(\beta) \in N(X, n) &\Leftrightarrow d_X(h(\beta), r_{((n)_1)_0}^X) < \frac{((n)_1)_1}{((n)_1)_2+1} \\ &\Leftrightarrow \exists m, i, l \in \omega \text{ Seq}(m) \wedge \text{lh}(m) = 2^i \wedge \beta|_i = s_l^i \wedge (m)_l = ((n)_1)_0 \wedge \\ &\quad d(h, h_m) < \frac{((n)_1)_1}{((n)_1)_2+1}, \end{aligned}$$

which gives the result. \square

- We then strengthen 7A.3 in [M] about **primitive recursion** as follows. If Z, Y are recursively presented Polish spaces, $g : Z \rightarrow Y$ and $h : Y \times \omega \times Z \rightarrow Y$ are Π_1^1 -recursive and $f : \omega \times Z \rightarrow Y$ is defined by

$$\begin{cases} f(0, z) := g(z), \\ f(n+1, z) := h(f(n, z), n, z), \end{cases}$$

then f is also Π_1^1 -recursive. If $m : Z \rightarrow Z$ is Π_1^1 -recursive, then the proof of 7A.3 in [M] shows that the map $f' : \omega \times Z \rightarrow Y$ defined by

$$\begin{cases} f'(0, z) := g(z), \\ f'(n+1, z) := h(f'(n, m(z)), n, z), \end{cases}$$

is also Π_1^1 -recursive. As in 7A.5 in [M], this can be extended to partial functions which are Π_1^1 -recursive on their domain.

- We are ready for the proof of our main tool.

Proof of Theorem 5.3. 3E.6 in [M] provides $\pi : \omega^\omega \rightarrow X$ recursive, $\mathcal{F} \in \Pi_1^0(\omega^\omega)$ and a Δ_1^1 -recursive injection $\rho : X \rightarrow \omega^\omega$ such that $\pi|_{\mathcal{F}}$ is injective, $\pi[\mathcal{F}] = X$ and ρ is the inverse of $\pi|_{\mathcal{F}}$. Let us show that the map $\mu : h \mapsto \pi \circ h$ is Δ_1^1 -recursive from $\mathcal{C}(2^\omega, \omega^\omega)$ into $\mathcal{C}(2^\omega, X)$. More generally, let Y be a recursively presented Polish space, and $\psi : Y \rightarrow \mathcal{C}(2^\omega, X)$. Note that

$$\begin{aligned} \psi(y) \in N(\mathcal{C}(2^\omega, X), n) &\Leftrightarrow d(\psi(y), h_{((n)_1)_0}) < \frac{((n)_1)_1}{((n)_1)_2+1} \\ &\Leftrightarrow \exists m \in \omega \sup_{\beta \in 2^\omega} d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) < \frac{((m)_1)_1}{((m)_1)_2+1} < \frac{((n)_1)_1}{((n)_1)_2+1} \\ &\Leftrightarrow \exists m \in \omega \forall \beta \in 2^\omega d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) < \frac{((m)_1)_1}{((m)_1)_2+1} < \frac{((n)_1)_1}{((n)_1)_2+1} \end{aligned}$$

and $h_{((n)_1)_0}(\beta) = r_{g(n, \beta)}^X$ for some recursive map $g : \omega \times 2^\omega \rightarrow \omega$.

In the present case, $Y = \mathcal{C}(2^\omega, \omega^\omega)$ and $\psi(y)(\beta) = \pi(y(\beta))$. Thus

$$\begin{aligned} d_X(\psi(y)(\beta), h_{((n)_1)_0}(\beta)) &< \frac{((m)_1)_1}{((m)_1)_2+1} \Leftrightarrow d_X(\pi(y(\beta)), r_{g(n,\beta)}^X) < \frac{((m)_1)_1}{((m)_1)_2+1} \\ &\Leftrightarrow \pi(y(\beta)) \in N(X, \langle 0, < g(n, \beta), ((m)_1)_1, ((m)_1)_2 \rangle) \\ &\Leftrightarrow (y(\beta), \langle 0, < g(n, \beta), ((m)_1)_1, ((m)_1)_2 \rangle) \in G^\pi, \end{aligned}$$

where G^π is the Σ_1^0 neighborhood diagram of π . As the evaluation map is recursive, $h \mapsto \pi \circ h$ is Π_1^1 -recursive and total, and thus Δ_1^1 -recursive.

• Let us show that there is a Δ_1^1 -recursive map $f: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X)$ such that $\mathcal{U}_\alpha^{\Pi_n^1, 2^\omega} = (f(\alpha))^{-1}(C)$ for each $\alpha \in \omega^\omega$. We adapt the proof of the main result in [K3]. We set $A := \pi^{-1}(C)$. As $C \in \Pi_n^1(X)$, $A \in \Pi_n^1(\omega^\omega)$. If $\langle \beta^0, \delta^0 \rangle \in 2^\omega$, then we inductively define, for $i \in \omega$, $m_i, \beta^{i+1}, \delta^{i+1}$ as follows. If (β^i, δ^i) is given and in $\text{Domain}(e)$, then $(\beta^i)^*(\delta^i) = \langle x_i, \beta^{i+1}, \delta^{i+1} \rangle$ and

$$m_i := \begin{cases} \text{the location of the first 0 in } x_i \text{ if it exists,} \\ 2 \text{ otherwise.} \end{cases}$$

We then set $Q := \{(\alpha, \langle \beta^0, \delta^0 \rangle) \in \omega^\omega \times 2^\omega \mid \forall i \in \omega \ (\beta^i, \delta^i) \in \text{Domain}(e) \wedge (\alpha, (m_i)) \in \mathcal{U}_\alpha^{\Pi_n^1, 2^\omega}\}$ and $B^* := Q_\alpha$, so that $Q \in \Pi_n^1(\omega^\omega \times 2^\omega)$ and $\beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q$ for each $(\alpha, \beta) \in \omega^\omega \times 2^\omega$ (note that B^* depends on α , but we denote it like this to keep the notation of [K3]). We define $I: \omega^\omega \rightarrow 2^\omega$ by $I(\alpha) := 0^{\alpha(0)} 10^{\alpha(1)} 1 \dots$. Note that I a Δ_1^1 -recursive injection onto the Π_2^0 set

$$\mathbb{P}_\infty := \{\beta \in 2^\omega \mid \forall p \in \omega \ \exists q \geq p \ \beta(q) = 1\},$$

so that there is a Δ_1^1 -recursive map $\phi: 2^\omega \rightarrow \omega^\omega$ which is the inverse of I on \mathbb{P}_∞ . We set

$$Q' := \left\{ \delta \in 2^\omega \mid (\delta)_0 \in \mathbb{P}_\infty \wedge \left(\phi((\delta)_0), (\delta)_1 \right) \in Q \right\},$$

so that $Q' \in \Pi_n^1(2^\omega)$. As $\mathcal{U}_\alpha^{\Pi_n^1, 2^\omega}$ is suitable, there is $\alpha_Q \in \omega^\omega$ recursive with $Q' = \mathcal{U}_{\alpha_Q}^{\Pi_n^1, 2^\omega}$. Note that

$$\begin{aligned} \beta \in B^* &\Leftrightarrow (\alpha, \beta) \in Q \Leftrightarrow \langle I(\alpha), \beta \rangle \in Q' \Leftrightarrow (\alpha_Q, \langle I(\alpha), \beta \rangle) \in \mathcal{U}_{\alpha_Q}^{\Pi_n^1, 2^\omega} \\ &\Leftrightarrow b(\mathcal{R}(\alpha_Q, \langle I(\alpha), \beta \rangle)) \in C \Leftrightarrow \rho(b(\mathcal{R}(\alpha_Q, \langle I(\alpha), \beta \rangle))) \in A. \end{aligned}$$

We set $G := \rho(b(\mathcal{R}(\alpha_Q, \langle I(\alpha), . \rangle)))$, so that $G: 2^\omega \rightarrow \omega^\omega$ is $\Delta_1^1(\alpha)$ -recursive and $\langle \beta^0, \delta^0 \rangle$ is in B^* if and only if $G(\langle \beta^0, \delta^0 \rangle) \in A$.

• As in [K3], we can find $F: 2^{<\omega} \rightarrow (2^\omega \times \omega)^{<\omega}$ satisfying the following properties:

- (1) $t \subseteq t' \Rightarrow F(t) \subseteq F(t')$
- (2) $|F(t)| = |t| + 1$
- (3) (i) if $F(\emptyset) = (\beta^0, k_0)$, then $\beta^0 \in \mathcal{D} \wedge \forall \delta^0 \in [\mathcal{O}(\beta^0)] \ G(\langle \beta^0, \delta^0 \rangle)(0) = k_0$
(ii) if $F(\varepsilon_0, \dots, \varepsilon_n) = (\beta^0, k_0, \beta^1, k_1, \dots, \beta^{n+1}, k_{n+1})$, then
(a) $\forall i \leq n+1 \ \beta^i \in \mathcal{D}$
(b) for all $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$, if $\delta^n, \dots, \delta^0$ are the uniquely determined members of $[\mathcal{O}(\beta^n)], \dots, [\mathcal{O}(\beta^0)]$ such that $\forall i \leq n \ (\beta^i)^*(\delta^i) = \langle \bar{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} \rangle$, where $\bar{\varepsilon}_i = 1^{\varepsilon_i} 0 1^\infty$, then $\forall i \leq n+1 \ G(\langle \beta^0, \delta^0 \rangle)(i) = k_i$.

We will need an effective version of this, so that we give the details of the construction of F . In fact, the β^i 's involved in the definition of F can be $\Delta_1^1(\alpha)$. In order to see this, we first define

$$H_0 : 2^\omega \times 2^\omega \rightarrow \omega$$

by $H_0(\beta, \delta) := G(< \beta, \delta >)(0)$. As G is $\Delta_1^1(\alpha)$ -recursive, H_0 too, and the niceness of the coding system gives $\beta^0 \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k_0 \in \omega$ such that $G(< \beta^0, \delta^0 >)(0) = k_0$ for each $\delta^0 \in [\mathcal{O}(\beta^0)]$. Now suppose that $n \in \omega$, $(\varepsilon_0, \dots, \varepsilon_n)$ and $F(\varepsilon_0, \dots, \varepsilon_{n-1}) = (\beta^0, k_0, \dots, \beta^n, k_n)$ are given. We define

$$H_{n+1} : 2^\omega \times 2^\omega \rightarrow \omega$$

as follows. Given $(\beta, \delta) \in 2^\omega \times 2^\omega$, let $\delta^n, \dots, \delta^0$ be the uniquely determined members of $[\mathcal{O}(\beta^n)], \dots, [\mathcal{O}(\beta^0)]$ resp., such that $(\beta^n)^*(\delta^n) = < \overline{\varepsilon}_n, \beta, \delta >$, and $(\beta^i)^*(\delta^i) = < \overline{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} >$ if $i < n$. Put $H_{n+1}(\beta, \delta) := G(< \beta^0, \delta^0 >)(n+1)$. As H_{n+1} is $\Delta_1^1(\alpha)$ (it is total and $\Pi_1^1(\alpha)$ -recursive since ι is Π_1^1 -recursive), the niceness of the coding system gives $\beta^{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k_{n+1} \in \omega$ such that $G(< \beta^0, \delta^0 >)(n+1) = k_{n+1}$ for each $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$. Then

$$F(\varepsilon_0, \dots, \varepsilon_n) := (\beta^0, k_0, \dots, \beta^{n+1}, k_{n+1}),$$

so that F is as desired. So we can assume that the β^i 's are $\Delta_1^1(\alpha)$ in the conditions required for F .

• By [K3] again, the map $h_\alpha : (\varepsilon_i) \mapsto (k_i)$ is continuous and $\mathcal{U}_\alpha^{\Pi_1^1, 2^\omega} = h_\alpha^{-1}(A)$. As this is not too long to prove, we give the details for completeness. The map h_α is in fact more than continuous: it is Lipschitz, by definition. Fix (ε_i) . We apply F to the initial segments of (ε_i) , which gives (β^i) . For each n , we define perfect sets $C_0^n, C_1^n, \dots, C_n^n \subseteq 2^\omega$ with $C_i^n \subseteq [\mathcal{O}(\beta^i)]$ if $i \leq n$, as follows:

$$\begin{aligned} C_n^n &:= \{\delta^n \in [\mathcal{O}(\beta^n)] \mid \exists \delta^{n+1} \in 2^\omega \text{ } (\beta^n)^*(\delta^n) = < \overline{\varepsilon}_n, \beta^{n+1}, \delta^{n+1} >\}, \\ C_{n-1}^n &:= \{\delta^{n-1} \in [\mathcal{O}(\beta^{n-1})] \mid \exists \delta^n \in C_n^n \text{ } (\beta^{n-1})^*(\delta^{n-1}) = < \overline{\varepsilon}_{n-1}, \beta^n, \delta^n >\}, \\ &\dots \\ C_0^n &:= \{\delta^0 \in [\mathcal{O}(\beta^0)] \mid \exists \delta^1 \in C_1^n \text{ } (\beta^0)^*(\delta^0) = < \overline{\varepsilon}_0, \beta^1, \delta^1 >\}. \end{aligned}$$

Note that

(4) $\delta^0 \in C_0^n \Rightarrow < \beta^i, \delta^i > \in \text{Domain}(e)$ for each $i \leq n$, where $\delta^1, \dots, \delta^n$ are computed according to the formula in (3).(ii).(b),

(5) $n' \geq n \Rightarrow \forall i \leq n \text{ } C_i^{n'} \subseteq C_i^n$.

This implies that $[\mathcal{O}(\beta^0)] \supseteq C_0^0 \supseteq C_0^1 \supseteq C_0^2 \supseteq \dots$ and $\bigcap_{n \in \omega} C_0^n$ contains some δ^0 . Note that $< \beta^i, \delta^i >$ is in $\text{Domain}(e)$, and $(\beta^i)^*(\delta^i) = < \overline{\varepsilon}_i, \beta^{i+1}, \delta^{i+1} >$ for each $i \in \omega$. By (3).(ii).(b),

$$G(< \beta^0, \delta^0 >) = k_i$$

for each $i \in \omega$. As $< \beta^0, \delta^0 > \in B^* \Leftrightarrow G(< \beta^0, \delta^0 >) \in A$,

$$(\forall i \in \omega \text{ } < \beta^i, \delta^i > \in \text{Domain}(e) \wedge (\varepsilon_i) \in \mathcal{U}_\alpha^{\Pi_1^1, 2^\omega}) \Leftrightarrow (k_i) \in A.$$

As $< \beta^i, \delta^i >$ is in $\text{Domain}(e)$ for each $i \in \omega$, $(\varepsilon_i) \in \mathcal{U}_\alpha^{\Pi_1^1, 2^\omega} \Leftrightarrow h_\alpha((\varepsilon_i)) = (k_i) \in A$.

• So we found, for each $\alpha \in \omega^\omega$, $h_\alpha \in \mathcal{C}(2^\omega, \omega^\omega)$ such that $\mathcal{U}_\alpha^{\Pi_1^1, 2^\omega} = (\pi \circ h_\alpha)^{-1}(C) = (\mu(h_\alpha))^{-1}(C)$. It remains to see that the map $\psi : \alpha \mapsto h_\alpha$, from ω^ω into $\mathcal{C}(2^\omega, \omega^\omega)$, can be Δ_1^1 -recursive (then f will be $\mu \circ \psi$). By the previous discussion, it is enough to see that the relation “ $k_i = k$ ” is Δ_1^1 in $(\alpha, (\varepsilon_i), i, k) \in \omega^\omega \times 2^\omega \times \omega^2$.

• We will define, by primitive recursion, a Δ_1^1 -recursive map $\tilde{f} : \omega \times \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$ such that $\tilde{f}(n, \alpha, (\varepsilon_i))$ will be of the form $(\langle \tilde{\beta}^0, \dots, \tilde{\beta}^n, \tilde{\beta}^n, \dots \rangle, \langle \tilde{k}^0, \dots, \tilde{k}^n \rangle)$ and can play the role of $F(\varepsilon_0, \dots, \varepsilon_{n-1})$. We first set

$$P := \left\{ (\alpha, (\varepsilon_i), \beta, k) \in \omega^\omega \times (2^\omega)^2 \times \omega \mid \right. \\ \left. \forall i \in \omega \ (\beta)_i = (\beta)_0 \in \mathcal{D} \cap \Delta_1^1(\alpha) \wedge \forall \delta \in [\mathcal{O}((\beta)_0)] \ G(\langle (\beta)_0, \delta \rangle)(0) = k \right\}.$$

Note that P is Π_1^1 and for any $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$ there is $(\beta, k) \in 2^\omega \times \omega$ such that $(\alpha, (\varepsilon_i), \beta, k) \in P$. The uniformization lemma gives a Δ_1^1 -recursive map $\tilde{g} : \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$ such that

$$(\alpha, (\varepsilon_i), \tilde{g}(\alpha, (\varepsilon_i))) \in P$$

for each $(\alpha, (\varepsilon_i)) \in \omega^\omega \times 2^\omega$. Then we set

$$D := \left\{ (\beta, p, n, \alpha, (\varepsilon_i)) \in 2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega \mid \text{Seq}(p) \wedge \text{lh}(p) = n+1 \wedge \forall q \in \omega \ (\beta)_q \in \mathcal{D} \cap \Delta_1^1(\alpha) \right\}.$$

Note that D is Π_1^1 , as well as

$$R := \left\{ (\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in D \times 2^\omega \times \omega \mid \forall i > n \ (\beta')_i = (\beta')_{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha) \wedge \right. \\ \left. \text{Seq}(k') \wedge \text{lh}(k') = n+2 \wedge \forall i \leq n \ (\beta')_i = (\beta)_i \wedge (k')_i = (p)_i \wedge \right. \\ \left. \forall \delta \in 2^\omega \left(\exists i \leq n+1 \ (\delta)_i \notin [\mathcal{O}((\beta')_i)] \vee \exists i \leq n \ (\beta')_i^*((\delta)_i) \neq \langle \bar{\varepsilon}_i, (\beta')_{i+1}, (\delta)_{i+1} \rangle \vee \right. \right. \\ \left. \left. \forall i \leq n+1 \ G(\langle (\beta')_0, (\delta)_0 \rangle)(i) = (k')_i \right) \right\}.$$

Moreover, for each $(\beta, p, n, \alpha, (\varepsilon_i)) \in D = \Pi_{2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega}[R]$ there is $(\beta', k') \in (2^\omega \cap \Delta_1^1(\alpha)) \times \omega$ such that $(\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in R$. The uniformization lemma gives a partial map

$$\tilde{h} : 2^\omega \times \omega^2 \times \omega^\omega \times 2^\omega \rightarrow 2^\omega \times \omega$$

which is Π_1^1 -recursive on its domain D , and such that $(\beta, p, n, \alpha, (\varepsilon_i), \tilde{h}(\beta, p, n, \alpha, (\varepsilon_i))) \in R$ if $(\beta, p, n, \alpha, (\varepsilon_i)) \in D$. This implies that the partial map \tilde{f} defined by

$$\begin{cases} \tilde{f}(0, \alpha, (\varepsilon_i)) := \tilde{g}(\alpha, (\varepsilon_i)), \\ \tilde{f}(n+1, \alpha, (\varepsilon_i)) := \tilde{h}(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)), \end{cases}$$

is Π_1^1 -recursive.

Moreover, an induction shows that $(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)) \in D$ for each $(n, \alpha, (\varepsilon_i))$, so that \tilde{f} is in fact total, and thus Δ_1^1 -recursive. More precisely, $\tilde{f}(n, \alpha, (\varepsilon_i))$ is of the form

$$(< \beta^0, \dots, \beta^n, \beta^n, \dots >, < k_0, \dots, k_n >),$$

where $(\varepsilon_0, \dots, \varepsilon_{n-1}) \mapsto (\beta^0, k_0, \dots, \beta^n, k_n)$ satisfies the properties (1)-(3) of F . It remains to note that $k_i = \tilde{f}(i, \alpha, (\varepsilon_i))(1)(i)$. \square

- We now prove the consequences of our main tool.

Definition 5.8 Let Γ be a class of subsets of recursively presented Polish spaces, $\mathbf{\Gamma}$ be the corresponding boldface class, X, Y be recursively presented Polish spaces, and $\mathcal{U} \in \Gamma(Y \times X)$. We say that \mathcal{U} is **effectively uniformly Y -universal for the $\mathbf{\Gamma}$ subsets of X** if the following hold:

- (1) $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$,
- (2) $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y \text{ } \Delta_1^1\text{-recursive}\}$,
- (3) for each $S \in \mathbf{\Gamma}(\omega^\omega \times X)$, there is a Borel map $b: \omega^\omega \rightarrow Y$ such that $S_\alpha = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^\omega$,
- (4) for each $S \in \mathbf{\Gamma}(\omega^\omega \times X)$, there is a Δ_1^1 -recursive map $b: \omega^\omega \rightarrow Y$ such that $S_\alpha = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^\omega$.

Notation. Let $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$ be a good ω^ω -universal for the Π_1^1 subsets of 2^ω , X_1 be a recursively presented Polish space, and \mathcal{C}_1 be a Π_1^1 subset of X_1 for which there is a Δ_1^1 -recursive map $b: \omega^\omega \rightarrow X_1$ such that

$$(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow b(< \alpha, \beta >) \in \mathcal{C}_1$$

if $(\alpha, \beta) \in \omega^\omega \times 2^\omega$. We define, for each natural number $n \geq 1$,

- $X_{n+1} := \mathcal{C}(2^\omega, X_n)$ (inductively),
- $\mathcal{C}_{n+1} := \{h \in X_{n+1} \mid \forall \beta \in 2^\omega \ h(\beta) \notin \mathcal{C}_n\}$ (inductively),
- $\mathcal{U}_n := \{(h, \beta) \in X_{n+1} \times 2^\omega \mid h(\beta) \in \mathcal{C}_n\}$.

Theorem 5.9 Let $n \geq 1$ be a natural number. Then

- (a) the set \mathcal{U}_n is effectively uniformly X_{n+1} -universal for the Π_n^1 subsets of 2^ω ,
- (b) the set \mathcal{C}_n is Π_n^1 -complete.

Proof. We argue by induction on n .

(a) Assume first that $n = 1$, and fix $S \in \Pi_1^1(\omega^\omega \times 2^\omega)$. Our assumption gives $b_1: \omega^\omega \rightarrow X_1$. As $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$ is a good ω^ω -universal for the Π_1^1 subsets of 2^ω , there is by Theorem 5.3 a Δ_1^1 -recursive map $f_1: \omega^\omega \rightarrow \mathcal{C}(2^\omega, X_1)$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow f_1(\alpha)(\beta) \in \mathcal{C}_1$ if $(\alpha, \beta) \in \omega^\omega \times 2^\omega$. Let $\alpha_S \in \omega^\omega$ with $S = \mathcal{U}_{\alpha_S}^{\Pi_1^1, \omega^\omega \times 2^\omega}$. Note that

$$(\alpha, \beta) \in S \Leftrightarrow (\mathcal{R}(\alpha_S, \alpha), \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow f_1(\mathcal{R}(\alpha_S, \alpha))(\beta) \in \mathcal{C}_1 \Leftrightarrow (f_1(\mathcal{R}(\alpha_S, \alpha)), \beta) \in \mathcal{U}_1.$$

As \mathcal{C}_1 is Π_1^1 , \mathcal{U}_1 too. If $A \in \Pi_1^1(2^\omega)$, then $A = \mathcal{U}_\alpha^{\Pi_1^1, 2^\omega}$ for some $\alpha \in \omega^\omega$. Applying the previous discussion to $S := \mathcal{U}^{\Pi_1^1, 2^\omega}$, we get $A = (\mathcal{U}_1)_{f_1(\mathcal{R}(\alpha_S, \alpha))}$, so that \mathcal{U}_1 is X_2 -universal for the Π_1^1 subsets of 2^ω , effectively and uniformly.

We now study \mathcal{U}_{n+1} . Fix $S \in \Pi_{n+1}^1(\omega^\omega \times 2^\omega)$. Let $\mathcal{U}^{\Pi_n^1, 2^\omega}$ be a good ω^ω -universal for the Π_n^1 subsets of 2^ω . We set $\mathcal{V}^{\Pi_{n+1}^1, 2^\omega} := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid \forall \delta \in 2^\omega \ (\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega}\}$, so that $\mathcal{V}^{\Pi_{n+1}^1, 2^\omega}$ is a suitable ω^ω -universal for the Π_{n+1}^1 subsets of 2^ω . Moreover, the induction assumption gives a Δ_1^1 -recursive map $b_{n+1}: \omega^\omega \rightarrow X_{n+1}$ such that

$$\begin{aligned} (\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} &\Leftrightarrow \forall \delta \in 2^\omega \ (\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow \forall \delta \in 2^\omega \ (b_{n+1}(\mathcal{R}(\alpha, \beta)), \delta) \notin \mathcal{U}_n \\ &\Leftrightarrow \forall \delta \in 2^\omega \ b_{n+1}(\mathcal{R}(\alpha, \beta))(\delta) \notin \mathcal{C}_n \Leftrightarrow b_{n+1}(\mathcal{R}(\alpha, \beta)) \in \mathcal{C}_{n+1} \end{aligned}$$

Theorem 5.3 gives a Δ_1^1 -recursive map f_{n+1} such that $(\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} \Leftrightarrow f_{n+1}(\alpha)(\beta) \in \mathcal{C}_{n+1}$ if $(\alpha, \beta) \in \omega^\omega \times 2^\omega$. Let

$$Q \in \Pi_n^1(\omega^\omega \times 2^\omega \times 2^\omega) \subseteq \Pi_n^1(\omega^\omega \times \omega^\omega \times 2^\omega)$$

such that $(\alpha, \beta) \in S \Leftrightarrow \forall \delta \in 2^\omega \ (\alpha, \beta, \delta) \notin Q$, and $\alpha_Q \in \omega^\omega$ such that $Q = \mathcal{U}_{\alpha_Q}^{\Pi_n^1, \omega^\omega \times \omega^\omega \times 2^\omega}$. Note that

$$\begin{aligned} (\alpha, \beta) \in S &\Leftrightarrow \forall \delta \in 2^\omega \ (\mathcal{R}(\mathcal{R}'(\alpha_Q, \alpha), \beta), \delta) \notin \mathcal{U}^{\Pi_n^1, 2^\omega} \Leftrightarrow (\mathcal{R}'(\alpha_Q, \alpha), \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^\omega} \\ &\Leftrightarrow f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha))(\beta) \in \mathcal{C}_{n+1} \Leftrightarrow (f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha)), \beta) \in \mathcal{U}_{n+1}. \end{aligned}$$

As $\mathcal{C}_n \in \Pi_n^1$, $\mathcal{C}_{n+1} \in \Pi_{n+1}^1$ and $\mathcal{U}_{n+1} \in \Pi_{n+1}^1$. If $A \in \Pi_{n+1}^1(2^\omega)$, then $A = \mathcal{U}_\alpha^{\Pi_{n+1}^1, 2^\omega}$ for some $\alpha \in \omega^\omega$. Applying the previous discussion to $S := \mathcal{U}^{\Pi_{n+1}^1, 2^\omega}$, we get $A = (\mathcal{U}_{n+1})_{f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha))}$, so that \mathcal{U}_{n+1} is X_{n+2} -universal for the analytic subsets of 2^ω , effectively and uniformly.

(b) By definition, $\mathcal{C}_1 \in \Pi_1^1$, and $\mathcal{C}_{n+1} \in \Pi_{n+1}^1$ if $\mathcal{C}_n \in \Pi_n^1$. Assume first that $E \in \Pi_n^1(2^\omega)$. Then $E = (\mathcal{U}_n)_h$ for some $h \in \mathcal{C}(2^\omega, X_n)$, by (a). Thus $E = h^{-1}(\mathcal{C}_n)$. If Z is a zero-dimensional Polish space and $D \in \Pi_n^1(Z)$, then we may assume that Z is a G_δ subset of 2^ω by 7.8 in [K2], so that $D \in \Pi_n^1(2^\omega)$. The previous discussion gives $g \in \mathcal{C}(2^\omega, X_n)$ with $D = g^{-1}(\mathcal{C}_n)$. Thus $D = (g|_Z)^{-1}(\mathcal{C}_n)$ and \mathcal{C}_n is Π_n^1 -complete. \square

Proof of Theorem 1.7. By Theorem 5.9, it is enough to show that if $\mathcal{U}^{\Pi_1^1, 2^\omega} \in \Pi_1^1$ is a good ω^ω -universal set for the Π_1^1 subsets of 2^ω , then there is a Δ_1^1 -recursive map $b: \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^\omega} \Leftrightarrow b(<\alpha, \beta>) \in \mathcal{P}$ if $(\alpha, \beta) \in \omega^\omega \times 2^\omega$. Let $H \in \Pi_2^0(\omega^\omega \times 2^\omega \times 2^\omega)$ such that $\neg \mathcal{U}^{\Pi_1^1, 2^\omega} = \Pi_{\omega^\omega \times 2^\omega}[H]$. We set $G := \{(\alpha, \beta) \in \omega^\omega \times 2^\omega \mid ((\alpha)_0, (\alpha)_1, (\beta)_1) \in H \wedge \beta \in \mathcal{K}\}$, so that $G \in \Delta_1^1(\omega^\omega \times 2^\omega)$, has G_δ vertical sections and $G \subseteq \omega^\omega \times \mathcal{K}$. Lemma 3.10 gives a Δ_1^1 -recursive map $F: \omega^\omega \rightarrow [0, 1]^{2^{<\omega}}$, taking values in \mathcal{M} , and such that $G_\alpha = \mathcal{V}_{b(\alpha)}$ for each $\alpha \in \omega^\omega$. If $(\alpha, \beta) \in \omega^\omega \times 2^\omega$, then

$$\begin{aligned} (\alpha, \beta) \notin \mathcal{U}^{\Pi_1^1, 2^\omega} &\Leftrightarrow \exists \delta \in 2^\omega \ (\alpha, \beta, \delta) \in H \Leftrightarrow \exists \delta \in 2^\omega \ (<\alpha, \beta>, \delta) \in G \\ &\Leftrightarrow \exists \delta \in 2^\omega \ (b(<\alpha, \beta>), \delta) \in \mathcal{V} \Leftrightarrow b(<\alpha, \beta>) \notin \mathcal{P}. \end{aligned}$$

This finishes the proof. \square

Questions. Let U be a Π_2^0 subset of $\omega^\omega \times 2^\omega$ which is universal for $\Pi_2^0(2^\omega)$. We set

$$G := \{(\alpha, \beta) \in \omega^\omega \times \mathcal{K} \mid (\alpha, (\beta)_1) \in U\}.$$

Note that G is a Π_2^0 subset of $\omega^\omega \times 2^\omega$ contained in $\omega^\omega \times \mathcal{K}$ which is universal for $\Pi_2^0(\mathcal{K})$. Indeed, fix $H \in \Pi_2^0(\mathcal{K})$. Then $H' := \{\gamma \in 2^\omega \mid <0^\infty, \gamma> \in H\}$ is Π_2^0 , which gives $\alpha_0 \in \omega^\omega$ with $H' = U_{\alpha_0}$. Then $H = G_{\alpha_0}$.

Let $\alpha \mapsto ((\alpha)_k)_{k \in \omega}$ be a homeomorphism between ω^ω and $(\omega^\omega)^\omega$, with inverse map

$$(\alpha_k)_{k \in \omega} \mapsto \langle \alpha_0, \alpha_1, \dots \rangle.$$

We set $S' := \{\alpha \in \omega^\omega \mid \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{(\alpha)_{\gamma(i)}}\}$. Note that S' is Σ_2^1 .

(1) *Is S' a Borel Σ_2^1 -complete set?*

Assume that this is the case. Then the set $\mathcal{S}_2 := \{(f_k)_{k \in \omega} \in \mathcal{M}^\omega \mid \exists \gamma \in \omega^\omega \ \forall i \in \omega \ f_{\gamma(i)} \in \mathcal{P}\}$ of sequences of martingales having a subsequence made of everywhere converging martingales is Borel Σ_2^1 -complete. Indeed, Lemma 3.10 gives a Borel map $F : \omega^\omega \rightarrow \mathcal{M}$ such that $G_\alpha = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^\omega$. The map $\tilde{F} : \omega^\omega \rightarrow \mathcal{M}^\omega$ defined by $\tilde{F}(\alpha)(k) := F((\alpha)_k)$ is Borel. Moreover,

$$\begin{aligned} \tilde{F}(\alpha) \in \mathcal{S}_2 &\Leftrightarrow \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin D\left(F((\alpha)_{\gamma(i)})\right) \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin \mathcal{V}_{F((\alpha)_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{(\alpha)_{\gamma(i)}} \\ &\Leftrightarrow \alpha \in S', \end{aligned}$$

so that $S' = \tilde{F}^{-1}(\mathcal{S}_2)$.

(2) *Is there a Borel map $f : \mathcal{C}(2^\omega, [0, 1]) \rightarrow \omega^\omega$ such that, for each $(h_k)_{k \in \omega} \in (\mathcal{C}(2^\omega, [0, 1]))^\omega$ and each $\beta \in 2^\omega$, the following are equivalent:*

- (a) $\lim_{k \rightarrow \infty} h_k(\beta) = 0$,
- (b) $\forall k \in \omega \ \beta \notin G_{f(h_k)}$?

Assume that this is the case. Then S' (and therefore \mathcal{S}_2) is Borel Σ_2^1 -complete, and thus Σ_2^1 -complete (see [P]). We define $F : (\mathcal{C}(2^\omega, [0, 1]))^\omega \rightarrow \omega^\omega$ by $F((h_k)_{k \in \omega}) := \langle f(h_0), f(h_1), \dots \rangle$, so that F is Borel. Note that

$$\begin{aligned} F((h_k)_{k \in \omega}) \in S' &\Leftrightarrow \exists \gamma \in \omega^\omega \ \forall i \in \omega \ \forall \beta \in 2^\omega \ \beta \notin G_{f(h_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma \in \omega^\omega \ \forall \beta \in 2^\omega \ \lim_{i \rightarrow \infty} h_{\gamma(i)}(\beta) = 0 \\ &\Leftrightarrow (h_k)_{k \in \omega} \in S, \end{aligned}$$

so that $S = F^{-1}(S')$.

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